Phase response curves for systems with time delay

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Summary. A phase response curve (PRC) is the main theoretical tool to analyze weakly perturbed limit cycle oscillators. It is well known how to compute phase response curves for systems described by ordinary differential equations. Systems with time delay can also demonstrate a stable limit cycle behavior, e.g., a Mackey-Glass system or chaotic systems stabilized by a time delay feedback control. We perform a phase reduction procedure for a time-delay system with the stable limit cycle and derive an equation for the PRC. An algorithm of PRC computation for time-delay systems is demonstrated by several specific examples.

Introduction

The complexity of time-delay systems is related to their infinite-dimensional phase space. However, under certain conditions, the phase space of time-delay system can be reduced and its dynamics can be modeled by a simple system of ordinary differential equations. Such a situation appears at bifurcation points. The most studied is a Hopf bifurcation, when the time-delay dynamics reduces to a normal form on a surface of a center manifold, which describes a birth of a small limit cycle [1]. Another situation, when a time-delay system admits a description by reduced equations, can appear far away from a bifurcation point. If a time-delay system has a stable limit cycle (it can be far away from the bifurcation point) and is disturbed by a small time-dependent perturbation, then its dynamics can be reduced to a phase dynamics on the limit cycle. The technique allowing such a reduction of systems described by ordinary differential equations (ODEs) is known as a phase reduction method [2]. Here we extend this method for time-delay systems.

Phase reduction of ordinary differential equations

First we recall the phase reduction method for ODEs. Consider a weakly perturbed limit cycle oscillator described by ODEs:

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}) + \varepsilon\mathbf{G}(t). \tag{1}$$

Here $\varepsilon\mathbf{G}(t)$ represent a small external signal. We suppose that for $\varepsilon = 0$ the system has a stable limit cycle with the period T: $\mathbf{X}(t) = \mathbf{X_c}(t) = \mathbf{X_c}(t-T)$. Then the equation for the phase φ of the perturbed system (1) reads [2]:

$$\dot{\varphi}(\varphi, t) = 1 + \varepsilon\mathbf{G}(t)\cdot\mathbf{Z}(\varphi), \tag{2}$$

where $\mathbf{Z}(\varphi)$ is the PRC. The PRC is a T periodic vector function that satisfies an adjoint equation:

$$\dot{\mathbf{Z}} = -D\mathbf{F}(\mathbf{X_c})^T\mathbf{Z} \tag{3}$$

with an initial condition $\mathbf{Z}(0)^T\dot{\mathbf{X}}_\mathbf{c}(0) = 1$. Here $D\mathbf{F}(\mathbf{X_c})$ is the Jacobian evaluated on the limit cycle. Note that Eq. (3) is unstable and its numerical solution is usually obtained via a backward integration. Since the Jacobian is unavailable in an analytical form its values are estimated from a forward integration of linearized equations.

Phase reduction of systems with time delay

Now we consider a weakly perturbed limit cycle oscillator described by a time-delay system:

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}(t), \mathbf{X}(t-\tau)) + \varepsilon\mathbf{G}(t). \tag{4}$$

Again, we suppose that for $\varepsilon = 0$ the system has a stable limit cycle: $\mathbf{X}(t) = \mathbf{X_c}(t) = \mathbf{X_c}(t-T)$. We managed to reduce this system to the phase equation (2) by two different methods, namely, (i) via a direct application of a phase reduction procedure to a time-delay system (4) and (ii) via approximation of the system (4) by ODEs and resort to the results of phase reduction theory from ODEs. Here we describe briefly the ideas of the second approach.
The time-delay system (4) can be approximated by ODEs as follows:

$$\begin{aligned}
\dot{\mathbf{X}}_0 &= \mathbf{F}(\mathbf{X}_0, \mathbf{X}_N) + \varepsilon\mathbf{G}(t),\\
\dot{\mathbf{X}}_1 &= \tfrac{N}{\tau}(\mathbf{X}_0 - \mathbf{X}_1),\\
&\vdots\\
\dot{\mathbf{X}}_N &= \tfrac{N}{\tau}(\mathbf{X}_{N-1} - \mathbf{X}_N).
\end{aligned} \tag{5}$$

If $N \to \infty$, then $\mathbf{X}_0(t) \to \mathbf{X}(t)$, $\mathbf{X}_N(t) \to \mathbf{X}(t-\tau)$, and solution of Eqs. (5) approaches the solution of Eq. (4). For any finite N, Egs. (5) represent an ODE system and we can apply the results described in the previous section. We linearize Eqs. (5), construct an adjoint and after taking the limit $N \to \infty$, obtain an equation for the PRC of time-delay system (4):

$$\dot{\mathbf{Z}} = -\mathbf{A}(t)^T\mathbf{Z}(t) - \mathbf{B}(t+\tau)^T\mathbf{Z}(t+\tau). \tag{6}$$

Here matrices $\mathbf{A}(t) = D_1\mathbf{F}(\mathbf{X_c}(t), \mathbf{X_c}(t-\tau))$, $\mathbf{B}(t) = D_2\mathbf{F}(\mathbf{X_c}(t), \mathbf{X_c}(t-\tau))$ and D_1 (D_2) denotes the vector derivatives with respect to the first (second) argument. The periodic solution of Eq. (6) can be obtained via a backward integration, since it is unstable as well as Eq. (3). The initial condition for Eq. (6) is:

$$\mathbf{Z}(0)^T\dot{\mathbf{X}}_\mathbf{c}(0) + \int_{-\tau}^{0} \mathbf{Z}(\tau+s)^T\mathbf{B}(\tau+s)\dot{\mathbf{X}}_\mathbf{c}(s)ds = 1. \tag{7}$$

Finally, the phase reduced system of a weakly perturbed time-delay oscillator (4) is defined by Eqs. (2), (6) and (7).

Example: Periodically perturbed phase-locked loop system

As an example of application of the above theory we consider a periodically perturbed phase-locked loop system [3]:

$$\dot{x} = -R\sin[x(t-1)] + \varepsilon G(t), \tag{8}$$

The parameter of the system are chosen such ($R = 2$) that for $\varepsilon = 0$ it displays a periodic oscillations with the period $T \approx 4$. The results of synchronization of the system with the sinusoidal $G(t) = \sin(2\pi\nu t)$ and rectangular $G(t) = \text{sign}[\sin(2\pi\nu t)]$ periodical signals are shown in Fig. 1. For a small frequency mismatch $\Delta\nu = 1/T - \nu$, the Arnold tongues obtained from the reduced system (2), (6)-(7) are in good agreement with those derived from the original time-delay system (8).

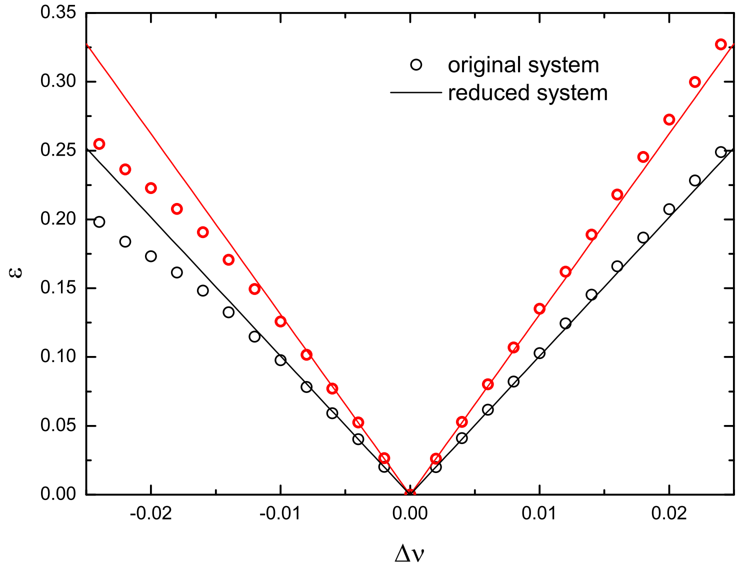

Figure 1: Arnold tongues (dependence of the threshold synchronization amplitude ε on the frequency mismatch $\Delta\nu = 1/T - \nu$) of system (8) perturbed by the sinusoidal (red) and rectangular (black) wave. Circles show the results of numerical simulation of the original time-delay system (8). Lines represent the results obtained from the reduced phase equation (2) with the PRC estimated from Eqs. (6)-(7)

Conclusions

We have considered the problem of weakly perturbed limit-cycle oscillations described by time-delay equations. By extending the phase reduction method to time-delay systems we have derived an equation for the phase response curve. The theoretical results are confirmed by numerical simulations of specific systems.

Acknowledgments

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