Time-delayed feedback control design beyond the odd-number limitation

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We present an algorithm for a time-delayed feedback control design to stabilize periodic orbits with an odd number of positive Floquet exponents in autonomous systems. Due to the so-called odd-number theorem such orbits have been considered as uncontrollable by time-delayed feedback methods. However, this theorem has been refuted by a counterexample and recently a corrected version of the theorem has been proved. In our algorithm, the control matrix is designed using a relationship between Floquet multipliers of the systems controlled by time-delayed and proportional feedback. The efficacy of the algorithm is demonstrated with the Lorenz and Chua systems.

I. INTRODUCTION

Control of complex and chaotic dynamics is one of the central issues in applied nonlinear science. Starting with the work of Ott *et al.* [1], a variety of methods have been developed in order to stabilize unstable periodic orbits (UPOs) embedded in a chaotic attractor by employing tiny control forces [2,3]. A particularly simple and efficient scheme is time-delayed feedback control (TDFC), introduced in Ref. [4] and later extended or modified by different authors, e.g., Ref. [5]. The TDFC has been successfully applied to many real-world problems in physical, chemical, and biological systems (see, e.g., Refs. [6] and [7] for reviews).

However, Nakajima [8] has pointed out that time-delayed feedback schemes suffer from the so-called odd-number limitation. Nakajima's theorem states that unstable periodic orbits with an odd number of real Floquet multipliers (FMs) larger than unity cannot be stabilized by time-delayed feedback control. In the main text of the paper [8], this limitation was proved for the case of nonautonomous systems, but it was also stated in the footnote that the same restriction should hold for the autonomous case. The limitation seemed to be supported by experimental and numerical evidence for both autonomous and nonautonomous systems, and over the following years research focused on a search for various modifications of the TDFC in order to bypass the limitation [9]. Significant knowledge has been gained 10 years after the publication of Nakajima's theorem, when Fiedler *et al.* [10] showed that the limitation is incorrect for autonomous systems. The authors of Ref. [10] considered a simple two-dimensional model system, a normal form for a subcritical Hopf bifurcation, which has a UPO with exactly one positive unstable Floquet multiplier, and showed that it can be stabilized by the conventional TDFC scheme (see also Ref. [11]). The mechanism of stabilization identified by Fiedler *et al.* has been shown to work close to a subcritical Hopf bifurcation in a Lorenz system [12] and in a laser experiment [13]. Similar results have been obtained for rotating waves near a fold bifurcation [14].

In all the examples above, the choice of the structure of the control matrix is strongly related to the fact that the system is close to a bifurcation point. Though the odd-number theorem is formally refuted for autonomous systems, there are no recipes for designing the control matrix far from bifurcation points. The aim of this paper is to fill this gap. Our research is mainly

encouraged by Hooton and Amann's recent publication [15], where the authors presented a corrected version of Nakajima's theorem for autonomous systems. We also use our recent results based on a phase reduction theory extended for systems with time delay [16,17] (see also Ref. [18]) and the ideas of Ref. [19], where an algorithm of reconstructing Floquet multipliers of the system controlled by time-delayed feedback from the system controlled by proportional feedback is described. Note that control design for controlling steady states of linear time-invariant systems with time delay has been considered in Ref. [20] (see also the references therein).

II. AN EXPLICIT FORM FOR HOOTON AND AMANN'S CRITERION

Let us consider an uncontrolled dynamical system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$ with $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ and assume that it has an unstable T-periodic orbit $\mathbf{x}(t) = \boldsymbol{\xi}(t) = \boldsymbol{\xi}(t + T)$, which we seek to stabilize by the time-delayed feedback control of the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{K}[\mathbf{x}(t - \tau) - \mathbf{x}(t)], \qquad (1)$$

where $\mathbf{K}$ is an $n \times n$ control matrix and τ is a positive delay time. Provided that the delay time coincides with the period of the orbit, $\tau = T$, the periodic solution $\boldsymbol{\xi}(t)$ of the free system is also a solution of Eq. (1) for any choice of the control matrix $\mathbf{K}$; i.e., the form (1) yields a noninvasive control scheme. A necessary condition for the stability of the solution $\boldsymbol{\xi}(t)$ of the controlled system (1) is given by Hooton and Amann's theorem [15]. To formulate this theorem let us assume that τ slightly differs from T. Then the controlled system (1) has a periodic solution close to $\boldsymbol{\xi}(t)$ with a new period Θ. Generally, the period Θ differs from τ and T; it is a function of $\mathbf{K}$ and τ, $\Theta = \Theta(\mathbf{K}, \tau)$, which satisfies $\Theta(\mathbf{K}, T) = T$. Hooton and Amann's theorem claims that the periodic solution $\boldsymbol{\xi}(t)$ is an unstable solution of the controlled system (1) if the condition

$$(-1)^m \lim_{\tau \to T} \frac{\tau - T}{\tau - \Theta(\mathbf{K}, \tau)} < 0 \qquad (2)$$

holds. Here m is a number of real Floquet multipliers larger than unity for the periodic solution $\boldsymbol{\xi}(t)$ of the uncontrolled system. The criterion (2) differs from Nakajima's version by the factor $\beta = \lim_{\tau \to T}(\tau - T)/(\tau - \Theta)$. It follows that the necessary (but not sufficient) condition for the TDFC to

stabilize a UPO with an odd number m is $\beta < 0$. This condition predicts correctly the location of the transcritical bifurcation, which provides successful stabilization of the UPO in the example of Fiedler *et al.* [10,15].

The criterion (2) can be rewritten in a more handy form. An explicit dependence of the factor β on the control matrix $\mathbf{K}$ can be derived from Eq. (2) by expanding Θ in terms of a small mismatch $\tau - T$ up to the second order. This problem has been solved in our recent paper [17] in a rather general formulation of a multiple-input multiple-output system (cf. Ref. [21] for the case of the scalar input). The approach used in Ref. [17] is based on a phase reduction theory extended for systems with time delay [16]. For the control law defined by Eq. (1), the result of Ref. [17] reads

$$\Theta(\mathbf{K},\tau) = T - (\tau - T)[\alpha(\mathbf{K}) - 1] + O[(\tau - T)^2], \quad (3)$$

where $\alpha(\mathbf{K})$ is a coefficient that relates the phase response curve (PRC) $\mathbf{z}(t)$ of the periodic orbit of the controlled system (for $\tau = T$) with the PRC $\boldsymbol{\rho}(t)$ of the same orbit of the uncontrolled system, $\mathbf{z}(t) = \alpha(\mathbf{K})\boldsymbol{\rho}(t)$. The latter expression shows that the profile of the PRC of the controlled orbit is independent of the control matrix $\mathbf{K}$; only its amplitude $\alpha(\mathbf{K})$ depends on $\mathbf{K}$. The PRC of the uncontrolled system can be computed as a T-periodic solution of the adjoint equation

$$\dot{\boldsymbol{\rho}}^T(t) = -\boldsymbol{\rho}^T(t) D\mathbf{f}(\boldsymbol{\xi}(t)), \quad (4)$$

for which the condition $\boldsymbol{\rho}^T(t)\dot{\boldsymbol{\xi}}(t) = 1$ holds for any t. Here the superscript T denotes the transpose operation and $D\mathbf{f}(\boldsymbol{\xi}(t))$ is the Jacobian matrix of the uncontrolled system estimated on the periodic orbit. By substituting Eq. (3) into Eq. (2) we obtain a simple relationship between the factor β and coefficient α: $\beta = \alpha^{-1}$. The coefficient α has been estimated in Ref. [17] so that for the factor β we get

$$\beta = \alpha^{-1}(\mathbf{K}) = 1 + \sum_{ij}^{n} K_{ij} C_{ij}, \quad (5)$$

where K_{ij} is the i,j element of the control matrix and

$$C_{ij} = \int_0^T \rho_i(t)\dot{\xi}_j(t) dt. \quad (6)$$

Here $\dot{\xi}_j(t)$ denotes the jth component of derivative of the periodic orbit and $\rho_i(t)$ is the ith component of the PRC of the uncontrolled orbit. Relation (5) expresses explicitly the dependence of the factor β on the control matrix. To compute the coefficients C_{ij} we need to solve Eq. (4). An algorithm for solution of this equation is described in Ref. [16]; it requires a knowledge of at least one control matrix that provides the successful stabilization of the target UPO. Below we describe another way of estimating the coefficients C_{ij}, without recourse to the solution of Eq. (4).

Note that the phase reduction theory identifies perfectly the transcritical bifurcation in the Fiedler *et al.* example [10]. When the delay-induced orbit coalesces with the target UPO the trivial Floquet multiplier $\mu = 1$ becomes degenerate. At the bifurcation point ($\beta = \alpha^{-1} = 0$) the amplitude α of the PRC $\mathbf{z}(t) = \alpha\boldsymbol{\rho}(t)$ of the controlled orbit tends to infinity; i.e., the phase of the system becomes extremely sensitive to external perturbations.

III. CONTROL MATRIX DESIGN

In what follows, we present a practical recipe for designing the control matrix when a target UPO of dynamical system has a single $m = 1$ real FM larger than unity. Any control matrix can be written in the form $\mathbf{K} = \kappa \tilde{\mathbf{K}}$, where κ is a scalar control gain and $\tilde{\mathbf{K}}$ is a matrix with at least one element equal to -1 or 1 and other elements in the interval $[-1,1]$. We can satisfy Hooton and Amann's necessary condition $\beta < 0$ for any given matrix $\tilde{\mathbf{K}}$ if we choose the control gain as

$$\kappa > \kappa^* \equiv -\left(\sum_{ij}^{n} \tilde{K}_{ij} C_{ij}\right)^{-1}. \quad (7)$$

However, this condition is not sufficient for the successful control. Without loss of the generality we assume that the threshold κ^* is positive, since this can be always achieved by appropriate choice of the sign of the matrix $\tilde{\mathbf{K}}$. We obtain additional conditions for $\tilde{\mathbf{K}}$ by using a relationship between the Floquet multipliers of the TDFC and proportional feedback control (PFC) systems [19]. Consider the PFC problem derived from Eq. (1) by replacing the time-delay term $\mathbf{x}(t - \tau)$ with $\boldsymbol{\xi}(t)$ and representing the control matrix as $\mathbf{K} = g\tilde{\mathbf{K}}$

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + g\tilde{\mathbf{K}}[\boldsymbol{\xi}(t) - \mathbf{x}(t)]. \quad (8)$$

The scalar g is a real-valued parameter that defines the feedback gain for the PFC system. The problem of stability of the periodic orbit controlled by proportional feedback is relatively simple. Small deviations $\delta\mathbf{x}(t) = \mathbf{x}(t) - \boldsymbol{\xi}(t)$ from the periodic orbit can be decomposed into eigenfunctions according to the Floquet theory $\delta\mathbf{x}(t) = \exp(\Lambda t)\mathbf{u}(t)$, where Λ is the Floquet exponent (FE) and the T-periodic Floquet eigenfunction $\mathbf{u}(t)$ satisfies

$$\dot{\mathbf{u}}(t) + \Lambda\mathbf{u}(t) = \left[D\mathbf{f}(\boldsymbol{\xi}(t)) - g\tilde{\mathbf{K}}\right]\mathbf{u}(t). \quad (9)$$

This equation produces n FEs Λ_j, $j = 1,\ldots,n$ [or FMs $\exp(\Lambda_j T)$]. The Floquet problem for the TDFC system (1) is considerably more difficult, since it is characterized by an infinite number of FEs. Let us denote the FEs of the periodic orbit controlled by time-delayed feedback by λ and the corresponding FMs by $\mu = \exp(\lambda T)$. The Floquet eigenvalue problem for the TDFC system can be presented in a form of Eq. (9) with the following replacement of the parameters: $\Lambda \to \lambda$ and $g \to \kappa[1 - \exp(-\lambda T)]$. Provided the FM $\exp(\lambda T)$ is real valued, this property leads to the following parametric equations (cf. Ref. [19]):

$$\lambda = \Lambda(g), \quad \kappa = g[1 - \exp(-\Lambda(g)T)]^{-1}, \quad (10)$$

which allow a simple reconstruction of the dependence $\lambda = \lambda(\kappa)$ for some of branches of FEs of the TDFC system using the knowledge of the similar dependence $\Lambda = \Lambda(g)$ for the PFC system. The dependence $\Lambda = \Lambda(g)$ is obtained by solving the Floquet problem (9). Though Eqs. (10) are valid only for the real-valued FMs, the examples below show that exactly these branches are most relevant for the stability of the TDFC system.

To demonstrate the advantages of Eqs. (10) we refer to the Lorenz system described by the state vector $\mathbf{x}(t) = $

$[x_1(t), x_2(t), x_3(t)]^T$ and the vector field [22]:

$$\mathbf{f}(\mathbf{x}) = [10(x_2 - x_1), x_1(28 - x_3) - x_2, x_1 x_2 - 8/3 x_3]^T. \quad (11)$$

We take the standard values of the parameters, which produce the classical chaotic Lorenz attractor, and consider the stabilization of its symmetric period-one UPO with the period $T \approx 1.559$ and the single unstable FM $\mu \approx 4.713$. In Fig. 1 we show three typical dependencies of the FEs on the coupling strength for the PFC $\Lambda = \Lambda(g)$ (left-hand column) and the TDFC $\lambda = \lambda(\kappa)$ (right-hand column) systems obtained with different matrices $\tilde{\mathbf{K}}$. The dependencies $\Lambda = \Lambda(g)$ for the PFC are derived from Eq. (9). We plot only two branches of the FEs, originated from the unstable FE of the free system (dashed curve) and the trivial FE (solid curve crossing the origin). The branch corresponding to the negative FE of the free system does not influence the stability of the TDFC. The dependencies $\lambda = \lambda(\kappa)$ for the TDFC are obtained using the transformation (10). We see that the case (a)–(b) provides successful control for the PFC but it is unsuccessful for the TDFC. The case (c)–(d) is again unsuccessful for the TDFC; here two real FEs coalesce in the positive region and produce a pair of complex conjugate FEs with the positive real part, which grows with the increase of κ. Finally, the case (e)–(f) is potentially successful for the TDFC; here the branch of unstable FE (which results from two branches of the PFC system) decreases monotonically with the increase of κ and becomes negative for $\kappa > \kappa^*$.

Let us consider in some details the properties of the transformation (10) at the singular point $g = \bar{g}$, where the trivial Λ_0 and the unstable Λ_u FEs of the PFC system coalesce, $\Lambda_0(\bar{g}) = \Lambda_u(\bar{g}) \equiv \bar{\Lambda}$. We note that Eqs. (10) guarantee the continuous differentiability of the function $\lambda(\kappa)$ at the point $\kappa = \bar{\kappa} \equiv \bar{g}[1 - \exp(-\bar{\Lambda}T)]^{-1}$, which results from the point $g = \bar{g}$ of the PFC system. Taking into account that $d\Lambda_0/dg$ and $d\Lambda_u/dg$ at the point $\bar{g}$ are infinite, we obtain from Eq. (10) that the left and right derivatives of $\lambda(\kappa)$ at $\kappa = \bar{\kappa}$ coincide, $[d\lambda/d\kappa]_{\kappa=\bar{\kappa}\pm 0} = -\bar{g}\exp(\bar{\Lambda}T)/(\bar{\kappa}^2 T)$.

Now we show that the threshold κ^* obtained from the FEs of the PFC system and transformation (10) coincides with the definition (7) derived from Hooton and Amann's criterion. The values $\lambda(\kappa)$ of the TDFC system with κ close to the threshold κ^* result from the values of the trivial FE $\Lambda_0(g)$ of the PFC system with g close to zero. To derive an expression for κ^* we expand the dependence $\Lambda_0(g)$ for the trivial FE in Taylor series

$$\Lambda_0(g)T = ag + bg^2 + O(g^3). \quad (12)$$

By substituting Eq. (12) into Eq. (10) and taking the limit $g \to 0$ we get $\kappa^* = a^{-1}$. An expression for the coefficient a can be derived by applying the perturbation theory to Eq. (9). To this end we write the trivial eigenmode in the form $\mathbf{u}(t) = \mathbf{u}_0(t) + g\mathbf{u}_1(t) + O(g^2)$. By substituting this expansion and Eq. (12) into Eq. (9), we get in zero approximation $\dot{\mathbf{u}}_0(t) = D\mathbf{f}(\boldsymbol{\xi}(t))\mathbf{u}_0(t)$. The solution of this equation is $\mathbf{u}_0(t) = \dot{\boldsymbol{\xi}}(t)$. In the first-order approximation, we obtain

$$\dot{\mathbf{u}}_1(t) = D\mathbf{f}(\boldsymbol{\xi}(t))\mathbf{u}_1(t) - (\tilde{\mathbf{K}} + \mathbf{I}a/T)\dot{\boldsymbol{\xi}}(t), \quad (13)$$

where $\mathbf{I}$ is the identity matrix. By multiplying Eq. (13) on the left-hand side (LHS) by $\boldsymbol{\rho}^T(t)$ and summing it with Eq. (4)
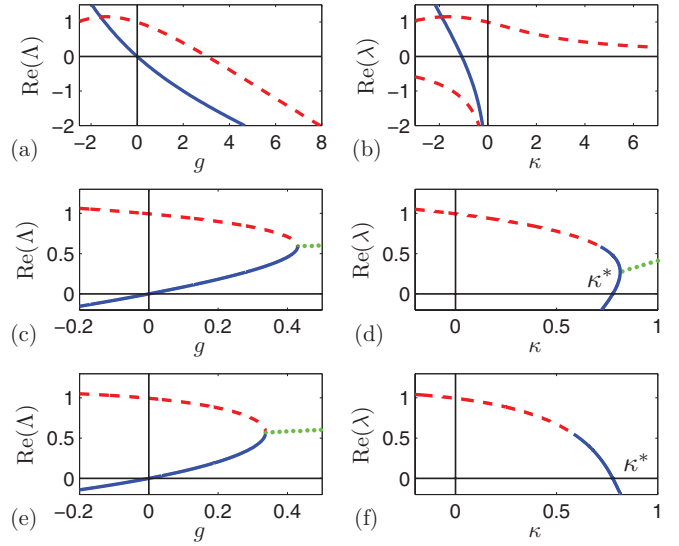

FIG. 1. (Color online) Three typical scenarios for the dependence of the FEs of the Lorenz system on the feedback gain for PFC (left-hand column) and TDFC (right-hand column) at different matrices $\tilde{\mathbf{K}}$: (a) and (b) $[0,0,0; 0,1,0; 0,0,0]$; (c) and (d) $[0,0,0; -1,0,0.3; 0,0,0]$; and (e) and (f) $[0,0,0; -1,0,0.5; 0,0,0]$. In panels (a), (c), and (e), solid (blue online) and dashed (red online) curves represent trivial and unstable FEs (both real-valued) for PFC, respectively. In panels (b), (d), and (f), the corresponding curves show reconstructed values of FEs for TDFC. κ^* is a threshold control gain, reconstructed from a segment of the trivial FE of the PFC system with g close to zero. Dotted (green online) curves in panels (c) and (e) show real parts of complex conjugate FEs, which cannot be transformed to TDFC by Eqs. (10). The dotted curve in panel (d) shows the real part of complex conjugate FEs emerged from coalescence of two real FEs of the TDFC system; it is computed by DDE-BIFTOOL [23].

multiplied on the right-hand side (RHS) by $\mathbf{u}_1(t)$, we get

$$\frac{d}{dt}(\boldsymbol{\rho}^T(t)\mathbf{u}_1(t)) = -\boldsymbol{\rho}^T(t)(\tilde{\mathbf{K}} + \mathbf{I}a/T)\dot{\boldsymbol{\xi}}(t). \quad (14)$$

Finally, we integrate this equation over the period T and obtain $a = -\int_0^T \boldsymbol{\rho}^T(t)\tilde{\mathbf{K}}\dot{\boldsymbol{\xi}}(t)dt$, which means that the value a^{-1} coincides with the threshold κ^* defined in Eq. (7).

A relation of the coefficient a with the matrix $\tilde{\mathbf{K}}$

$$a = -\sum_{ij}^n \tilde{K}_{ij} C_{ij} \quad (15)$$

provides an alternative way to estimate the coefficients C_{ij}. The particular coefficient $C_{\tilde{i}\tilde{j}}$ can be estimated as $C_{\tilde{i}\tilde{j}} = -a$ if we choose the matrix $\tilde{\mathbf{K}}$ with all zero elements except for $\tilde{K}_{\tilde{j}\tilde{i}} = 1$. Then the coefficient a in expansion (12) can be obtained by numerical computation of the dependence $\Lambda_0(g)$ for small g.

Apart from Hooton and Amann's condition (7), the successful control requires that the derivative $d\lambda/d\kappa$ at the threshold $\kappa = \kappa^*$ to be negative [see Fig. 1(f)]. By substituting Eq. (12) into Eq. (10) we get

$$\left.\frac{d\lambda}{d\kappa}\right|_{\kappa=\kappa^*} = \lim_{g\to 0}\frac{d\Lambda_0/dg}{d\kappa/dg} = \frac{2a}{T(1 - 2b/a^2)} < 0. \quad (16)$$

The parameter a is positive by assumption of the positiveness of κ^*. Then this condition simplifiers to

$$1 - 2b/a^2 < 0. \tag{17}$$

By extending the above perturbation theory for Eq. (9) up to the second-order terms with respect to g, we derive the following expression for the coefficient b:

$$b = -\frac{a}{T}\int_0^T \boldsymbol{\rho}^T(t)\mathbf{u}_1(t)dt - \int_0^T \boldsymbol{\rho}^T(t)\tilde{\mathbf{K}}\mathbf{u}_1(t)dt. \tag{18}$$

This allows us to write the relation of the coefficient b with the matrix $\tilde{\mathbf{K}}$ in the quadratic form

$$b = \sum_{ijkl}^n \tilde{K}_{ij}\tilde{K}_{kl}D_{ijkl} \tag{19}$$

with coefficients $D_{ijkl} = D_{klij}$. These coefficients can be obtained in a similar way as the coefficients C_{ij} by taking specific forms of the matrix $\tilde{\mathbf{K}}$ and estimating b from the dependence $\Lambda_0(g)$ of the trivial FE for small g.

The knowledge of the coefficients C_{ij} and D_{ijkl} allows an explicit computation of the parameters a and b for any given matrix $\tilde{\mathbf{K}}$. As a result we can simply verify the condition (17) and estimate the threshold κ^* in Eq. (7).

Finally, we can summarize our algorithm as follows: (i) choose the structure of the matrix $\tilde{\mathbf{K}}$ with only several nonzero elements in such a way as to make possible the coalescence of the positive and trivial Floquet branches of the PFC system [as in Figs. 1(c) or 1(e)]; (ii) for the given structure of the matrix $\tilde{\mathbf{K}}$, estimate the relevant coefficients C_{ij} and D_{ijkl}; (iii) choose the values of nonzero elements of the matrix $\tilde{\mathbf{K}}$ such as to satisfy condition (17); (iv) compute the threshold κ^* and satisfy condition (7).

We stress that our algorithm considers only the most important branches of the FEs and its final outcome has to be verified by more detailed analysis of the stability of the TDFC system. Nevertheless, the algorithm gives a simple practical recipe for the selection of appropriate control matrices and works well for typical chaotic systems.

Here we have presented an algorithm for the design of the control matrix $\tilde{\mathbf{K}}$. In control engineering, the control matrix is usually presented as a product of two matrices $\tilde{\mathbf{K}} = \mathbf{BG}$, where $\mathbf{B}$ is an $n \times k$ input matrix, and $\mathbf{G}$ is an $k \times n$ gain matrix. Here k denotes the dimension of the control variable, which is typically less than the system dimension, $k < n$. The matrix $\mathbf{B}$ relates the control variable with the system input and for the specific system is not free to choose, while the matrix $\mathbf{G}$ relates the state variable with the control variable and is free to choose. In such a formulation, the control design means the design of the gain matrix $\mathbf{G}$. Our algorithm can be simply reformulated for this case by substituting $\tilde{\mathbf{K}} = \mathbf{BG}$ in Eqs. (7) and (19) and treating the inequalities (7) and (17) as conditions for the coefficients of the gain matrix $\mathbf{G}$ provided the matrix $\mathbf{B}$ is given. Such an approach is actual when considering the specific experimental system with the given input law $\mathbf{B}$. We stress that our paper is focused on the principal question of whether is it possible to stabilize a UPO with an odd number limitation when the UPO is far from a bifurcation point. To make the presentation of this problem more transparent, in examples below we focus on design of the matrix $\tilde{\mathbf{K}}$ rather

than $\mathbf{G}$. For the given $\mathbf{B}$, the matrix $\mathbf{G}$ is related with the matrix $\tilde{\mathbf{K}}$ by a simple matrix equation $\tilde{\mathbf{K}} = \mathbf{BG}$.

IV. NUMERICAL EXAMPLES

First we discuss the details of application of our algorithm for the Lorenz system (11). Motivated by a "commonsense" assumption we started our analysis with the diagonal matrix $\tilde{\mathbf{K}}$. However, it appeared that such a choice, which works well for PFC systems, does not satisfy the first point of our algorithm. The impossibility to attain successful control with the diagonal control matrix can probably explain why the Lorenz system has not been stabilized by a conventional TDFC until now. We found that the requirements of our algorithm can be satisfied by many different nondiagonal configurations of the matrix $\tilde{\mathbf{K}}$. Here we show the results with the matrix $\tilde{\mathbf{K}}$ that has only
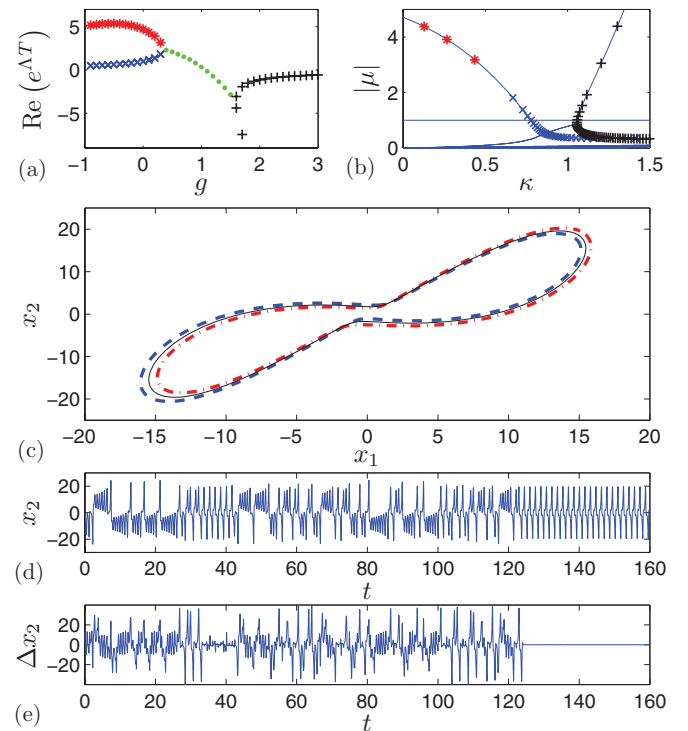

FIG. 2. (Color online) Stabilization of the period-one UPO of the Lorenz system with the matrix $\tilde{\mathbf{K}} = [0,0,0; -1,0,0.5; 0,0,0]$. (a) FMs vs g for the PFC. Crosses (blue online) and asterisks (red online) represent the trivial and unstable branches, respectively (they are real valued). Gray (green) points represent the real part of complex conjugate FMs. Black "plus signs" show a new pair of real-valued branches appeared from the complex conjugate FMs. (b) Absolute values of FMs vs κ for the TDFC. Solid curves are obtained by DDE-BIFTOOL, while symbols show the reconstruction of the FMs from panel (a) via Eqs. (10). Both branches (one marked by asterisks and crosses and another by "plus signs") that define the stability interval $\kappa \in [0.78,1.06]$ of the TDFC are reconstructed from the PFC system. (c) (x_1,x_2) projection of periodic orbits. Dashed (blue online) and dash-dotted (red online) curves show the stable delay-induced orbit for $\kappa = 0.63$ before the trascritical bifurcation ($\kappa^* \approx 0.78$) and the unstable delay-induced orbit for $\kappa = 1.05$ after the bifurcation, respectively. The target orbit is presented by the black solid curve. [(d) and (e)] Dynamics of $x_2(t)$ and difference $\Delta x_2(t) = x_2(t) - x_2(t - \tau)$ for $\kappa = 0.865$ and filter parameters $\tau_w = 0.5$ and $\varepsilon = 2$.
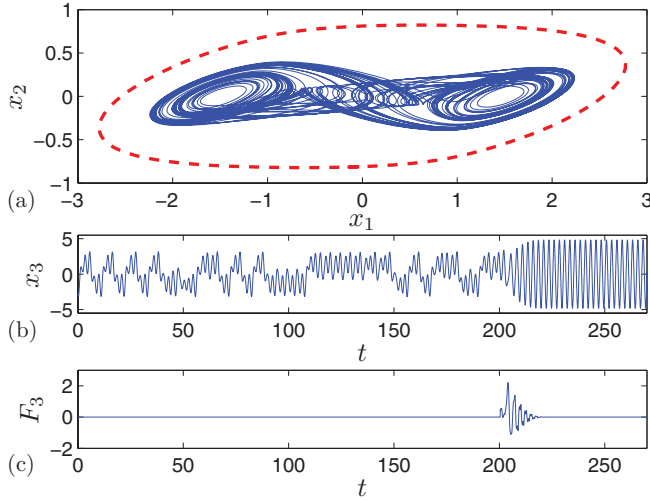
FIG. 3. (Color online) Stabilization of a UPO in the Chua system with $\tilde{\mathbf{K}} = [0,0,0; 0,0,0; 1,0,0.3]$. (a) The (x_1,x_2) projection of a chaotic attractor (solid curve, blue online) and the target UPO (dashed curve, red online). [(b) and (c)] Dynamics of $x_3(t)$ and the third component of the TDFC force $F_3(t) = \kappa\{\tilde{K}_{31}[x_1(t-\tau) - x_1(t)] + \tilde{K}_{33}[x_3(t-\tau) - x_3(t)]\}$. The control is switched on at $t = 200$ with $\kappa = 1.2$.

two nonzero elements $\tilde{K}_{21} = -1$ and $-1 \leqslant \tilde{K}_{23} \leqslant 1$. The relevant coefficients for such a matrix configuration are $C_{21} \approx 1.286$, $C_{23} \approx 1.5 \times 10^{-3}$, $D_{2121} \approx 0.163$, $D_{2323} \approx 3.792$, and $D_{2123} = D_{2321} \approx 9.7 \times 10^{-8}$. The inequality (17) leads to the requirement $|\tilde{K}_{23}| > 0.418$. We choose $\tilde{K}_{23} = 0.5$ and obtain the threshold $\kappa^* \approx 0.78$. As is seen from Fig. 2, these estimates predict correctly the successful control. In Figs. 2(a) and 2(b) we compare the values of FMs of the TDFC system reconstructed from the PFC system with those obtained via direct analysis of the TDFC system by the DDE-BIFTOOL package [23]. Surprisingly, Eqs. (10) allow us to obtain not only the threshold κ^* but also the interval of stability of the controlled orbit, since the branch of FMs (marked by "plus signs") that defines the loss of the stability is reconstructed from the PFC system as well. The stabilization of the UPO at the threshold κ^* is caused by a transcritical bifurcation as in the example of Fiedler *et al.* [10]. The delay-induced periodic orbits in the vicinity of the bifurcation point are shown in Fig. 2(c). Finally, Figs. 2(d) and 2(e) show the dynamics of the controlled system obtained by integration

of Eqs. (1) and (11) [24]. To reduce the transient time, the moment of switching on the control has been determined by a filter equation $\dot{w} = \{|x_1(t) - x_1(t - \tau)| - w(t)\}/\tau_w$ [25]. The filter estimates the closeness of the system state to the UPO and the control is activated only when the variable w becomes small, $w(t) < \varepsilon$.

To demonstrate the universality of our approach we refer to another example, the Chua system [26] defined by the state vector $\mathbf{x}(t) = [x_1(t),x_2(t),x_3(t)]^T$ and the vector field

$$\mathbf{f}(\mathbf{x}) = [9(x_2 - \phi(x_1)), x_1 - x_2 + x_3, -100/7 x_2]^T, \quad (20)$$

where $\phi(x_1) = 2/7 x_1 - 3/14(|x_1 + 1| - |x_1 - 1|)$. The (x_1,x_2) projection of a chaotic trajectory and the target UPO of the system are shown in Fig. 3(a). Here the target UPO is outside of the strange attractor; its period is $T \approx 2.483$ and the single unstable FM $\mu \approx 2.325$. We choose a nondiagonal configuration of the matrix $\tilde{\mathbf{K}}$ with two nonzero elements $\tilde{K}_{31} = 1$ and $-1 \leqslant \tilde{K}_{33} \leqslant 1$. Then the relevant coefficients are $C_{31} \approx -2.02$, $C_{33} \approx 3.01$, $D_{3131} \approx 2.46$, $D_{3333} \approx 1.85$, and $D_{3133} = D_{3331} \approx -2.21$. For $\tilde{K}_{33} = 0.3$, the inequality (17) is satisfied and the threshold value of the control gain is $\kappa^* \approx 0.89$. The successful stabilization of the UPO is demonstrated in Figs. 3(b) and 3(c) for $\kappa = 1.2$.

V. CONCLUSIONS

In conclusion, we have presented a practical recipe for a time-delayed feedback control design, which enables the stabilization of periodic orbits with an odd number of real Floquet multipliers larger than unity. The algorithm is suited for autonomous systems far from bifurcation points of periodic orbits. By way of examples we have designed appropriate control matrices for the Lorenz and Chua systems to stabilize their symmetric periodic orbits with one positive unstable Floquet multiplier. Before now these systems have been considered as classical examples which are inaccessible for the conventional time-delayed feedback control. We hope that our findings will enhance the ability of further implementations of time-delayed feedback control in practical applications.

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