CENTER FOR PHYSICAL SCIENCES AND TECHNOLOGY

CARL VON OSSIETZKY universität OLDENBURG

XXXI DYNAMICS DAYS EUROPE 2011

Phase response curve for systems with time delay

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Motivation

A phase response curve (PRC) is the main theoretical tool to analyze weakly perturbed limit cycle oscillators. It is well known how to compute phase response curves for systems described by ordinary differential equations. Systems with time delay can also demonstrate a stable limit cycle behavior, e.g., a Mackey-Glass system or chaotic systems stabilized by a time delay feedback control. We perform a phase reduction procedure for a time-delay system with the stable limit cycle and derive an equation for the phase response curve. An algorithm of PRC computation for time-delay systems is demonstrated by several specific examples.

Phase reduction

Ordinary differential equation | Time delay equation

Weakly perturbed limit cycle oscillator:

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}) + \varepsilon \mathbf{G}(t). \qquad \Big| \qquad \dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}(t), \mathbf{X}(t-\tau)) + \varepsilon \mathbf{G}(t).$$

The equation for the phase φ of the perturbed system reads:

$$\dot{\varphi}(\varphi, t) = 1 + \varepsilon \mathbf{G}(t) \cdot \mathbf{Z}(\varphi),$$

where $\mathbf{Z}(\varphi)$ is the phase response curve. The phase response curve is a periodic vector function that satisfies an adjoint equation:

$$\dot{\mathbf{Z}} = -D\mathbf{F}(\mathbf{X_c})^T \mathbf{Z}, \qquad \Big| \qquad \dot{\mathbf{Z}} = -\mathbf{A}(t)^T \mathbf{Z}(t) - \mathbf{B}(t+\tau)^T \mathbf{Z}(t+\tau),$$

here $\mathbf{X_c}$ is the limit cycle of the free system and matrices $\mathbf{A}(t) = D_1\mathbf{F}(\mathbf{X_c}(t), \mathbf{X_c}(t-\tau))$, $\mathbf{B}(t) = D_2\mathbf{F}(\mathbf{X_c}(t), \mathbf{X_c}(t-\tau))$ denote the vector derivatives with respect to the first and second argument.

An initial condition for the adjoint equation must satisfy:

$$\mathbf{Z}(0)^T \dot{\mathbf{X}}_\mathbf{c}(0) = 1. \qquad \Big| \qquad \mathbf{Z}(0)^T \dot{\mathbf{X}}_\mathbf{c}(0) + \int_{-\tau}^{0} \mathbf{Z}(\tau+s)^T \mathbf{B}(\tau+s) \dot{\mathbf{X}}_\mathbf{c}(s) ds = 1.$$

Example: Mackey-Glass system

$$\frac{dx}{dt} = \frac{2 \cdot x(t-0.7)}{1 + [x(t-0.7)]^{10}} - x(t) + \varepsilon g(t)$$

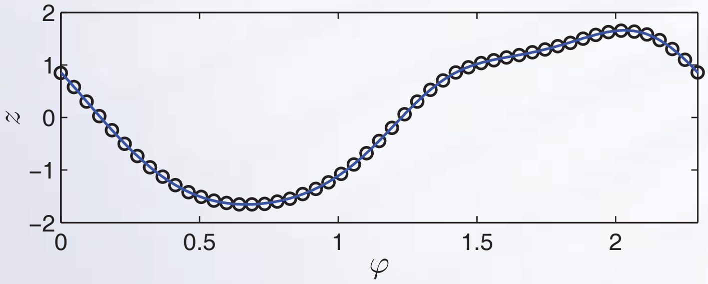

Figure 1: The PRC of the Mackey-Glass system. Circles show the value of PRC derived from system response to small δ-pulse and the solid curve represents the solution of the adjoint equation.

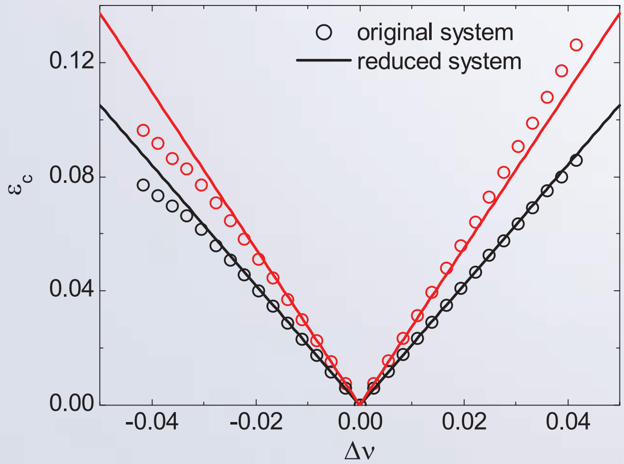

Figure 2: Arnold tongues (the dependence of the threshold synchronization amplitude on the frequency mismatch) of the Mackey-Glass system perturbed by the sinusoidal (red) and rectangular (black) wave. Circles show the results from original time-delay system, while lines represent the results from the phase equation.

Example: Rossler system stabilized by delayed feedback control (DFC) method

Chaotic system stabilized by DFC:

$$\dot{\mathbf{X}} = \mathbf{F}\left(\mathbf{X}(t)\right) + \mathbf{K}\left[\mathbf{X}(t-\tau) - \mathbf{X}(t)\right]$$

Since the delay time is equal to the system period, this leads to the simplest adjoint equation:

$$\dot{\mathbf{Z}} = -D\mathbf{F}\left(\mathbf{X_c}(t)\right)^T \mathbf{Z}(t)$$

The adjoint equation is independent of the control matrix $\mathbf{K}$. This means that the profile of the PRC is invariant with respect to the variation of $\mathbf{K}$.

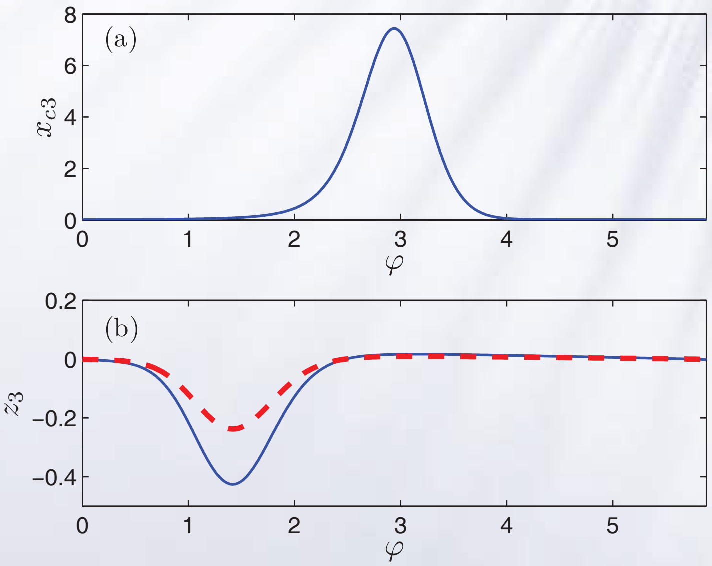

Figure 3: (a) The third component of the period-one UPO and (b) the third component of the PRC of the Rossler system subject to the DFC. The blue solid and red dashed curves correspond to the control gain K=0.15 and K= 0.5, respectively.