Analytical treatment of quantum systems driven by amplitude-modulated time-periodic force using flow equation approach

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Main task and motivation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi\left(t\right)\right\rangle = h\left(\omega t, t\right)\left|\psi\left(t\right)\right\rangle$$

periodic dependence with respect to the first argument $h\left(\omega t + 2\pi, t\right) = h\left(\omega t, t\right)$

$\hbar\omega \gg$ any other characteristic energies of the system, or in other words

$$h\left(\omega t, t\right) = \sum_{n=-\infty}^{+\infty} h^{(n)}\left(t\right)\mathrm{e}^{\mathrm{i}n\omega t}$$

matrix elements $\left|h_{\alpha\beta}^{(n)}\right| \ll \hbar\omega$

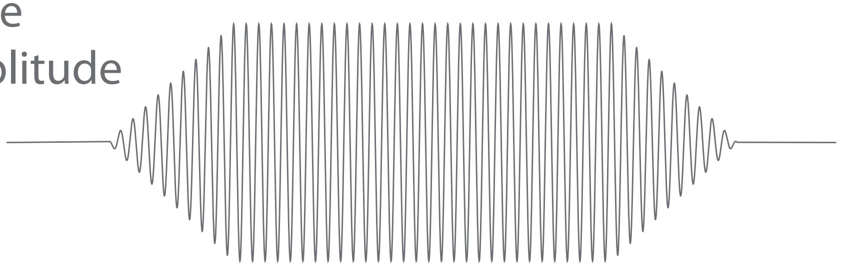
derivative of matrix elements $\left|\dot{h}_{\alpha\beta}^{(n)}\right| \ll \left|h_{\alpha\beta}^{(n)}\right|\omega$

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi\left(t\right)\right\rangle = h_{\mathrm{eff}}\left(t\right)\left|\psi\left(t\right)\right\rangle$$

V. Novičenko, E. Anisimovas, G. Juzeliūnas: *Phys. Rev. A* **95**, 023615 (2017)
V. Novičenko, G. Žlabys, E. Anisimovas: *Phys. Rev. A* **105**, 012203 (2022)

Motivation

Shaken optical lattice



R. Desbuquois, M. Messer, F. Görg, K. Sandholzer, G. Jotzu, T. Esslinger: *Phys. Rev. A* **96**, 053602 (2017)

Extension of the space

Let us study whole family of the solutions:

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi_\theta(t)\right\rangle = h(\omega t+\theta,t)\left|\psi_\theta(t)\right\rangle \qquad \theta\in[0,2\pi)$$

with initial conditions

$$\left|\psi_{\theta+2\pi}(t_{\mathrm{in}})\right\rangle = \left|\psi_\theta(t_{\mathrm{in}})\right\rangle$$

Hamiltonian $h(\omega t+\theta,t)$ acts on a Hilbert space $\mathscr{H}$

Introduce the space $\mathscr{T}$ of θ-periodic functions

Construct the space $\mathscr{L}=\mathscr{H}\otimes\mathscr{T}$

Apply unitary transformation $\mathcal{U}=\exp\left[\omega t\frac{\partial}{\partial\theta}\right]$

$$\mathcal{K}(t)=\mathcal{U}^\dagger\mathcal{H}(\omega t,t)\mathcal{U}-\mathrm{i}\hbar\mathcal{U}^\dagger\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t}=-\mathrm{i}\hbar\omega\frac{\partial}{\partial\theta}\otimes\mathbf{1}_{\mathscr{H}}+\sum_{n=-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}n\theta}\otimes h^{(n)}(t)$$

Orthonormal basis of the space $\mathscr{T}$

$$\frac{\mathrm{e}^{\mathrm{i}n\theta}}{\sqrt{2\pi}}\leftrightarrow|n\rangle$$

$$\mathcal{K}(t)=\sum_{n=-\infty}^{+\infty}|n\rangle\, n\hbar\omega\,\langle n|\otimes\mathbf{1}_{\mathscr{H}}+\sum_{n,m=-\infty}^{+\infty}|m\rangle\langle n|\otimes h^{(m-n)}(t)$$

"Kamiltonian" operator and effective Hamiltonian

$$\mathcal{K}(t) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & h^{(0)} - \hbar\omega & h^{(-1)} & h^{(-2)} & h^{(-3)} & \cdots \\ \cdots & h^{(1)} & h^{(0)} & h^{(-1)} & h^{(-2)} & \cdots \\ \cdots & h^{(2)} & h^{(1)} & h^{(0)} + \hbar\omega & h^{(-1)} & \cdots \\ \cdots & h^{(3)} & h^{(2)} & h^{(1)} & h^{(0)} + 2\hbar\omega & \cdots \\ \cdots & h^{(4)} & h^{(3)} & h^{(2)} & h^{(1)} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Apply unitary operator $\mathcal{D}(t)$ that block-diagonalize the Kamiltonian $\mathcal{K}_D(t) = \mathcal{D}^\dagger \mathcal{K} \mathcal{D} - \mathrm{i}\hbar \mathcal{D}^\dagger \frac{\mathrm{d}\mathcal{D}}{\mathrm{d}t}$

$$\mathcal{K}_D(t) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & h_{\text{eff}}(t) - \hbar\omega & \mathbf{0}_{\mathscr{H}} & \mathbf{0}_{\mathscr{H}} & \mathbf{0}_{\mathscr{H}} & \cdots \\ \cdots & \mathbf{0}_{\mathscr{H}} & h_{\text{eff}}(t) & \mathbf{0}_{\mathscr{H}} & \mathbf{0}_{\mathscr{H}} & \cdots \\ \cdots & \mathbf{0}_{\mathscr{H}} & \mathbf{0}_{\mathscr{H}} & h_{\text{eff}}(t) + \hbar\omega & \mathbf{0}_{\mathscr{H}} & \cdots \\ \cdots & \mathbf{0}_{\mathscr{H}} & \mathbf{0}_{\mathscr{H}} & \mathbf{0}_{\mathscr{H}} & h_{\text{eff}}(t) + 2\hbar\omega & \cdots \\ \cdots & \mathbf{0}_{\mathscr{H}} & \mathbf{0}_{\mathscr{H}} & \mathbf{0}_{\mathscr{H}} & \mathbf{0}_{\mathscr{H}} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Full dynamics of the state vector: $|\psi(t_{\text{fn}})\rangle = U_{\text{Micro}}(\omega t_{\text{fn}}, t_{\text{fn}}) \boxed{U_{\text{eff}}(t_{\text{fn}}, t_{\text{in}})} U^\dagger_{\text{Micro}}(\omega t_{\text{in}}, t_{\text{in}}) |\psi(t_{\text{in}})\rangle$

Flow towards diagonalization

Main idea is to gradually diagonalize the Hamiltonian.

run flow equation

$H(s)|_{s=0} =$ initial Hamiltonian $\longrightarrow$ $H(s)|_{s=+\infty} =$ fully diagonal Hamiltonian

For discrete flow:

$$H(s+1) = U(s)\, H(s)\, U^{\dagger}(s) = \mathrm{e}^{\eta(s)} H(s)\, \mathrm{e}^{-\eta(s)}$$

For continuous flow:

$$H(s+\mathrm{d}s) = \mathrm{e}^{\eta(s)\mathrm{d}s} H(s)\, \mathrm{e}^{-\eta(s)\mathrm{d}s} = H(s) + [\eta(s), H(s)]\,\mathrm{d}s + \mathcal{O}\left((\mathrm{d}s)^2\right)$$

$\downarrow$

$$\frac{\mathrm{d}H(s)}{\mathrm{d}s} = [\eta(s), H(s)]$$

The quantity $I_p = \mathrm{Tr}\left[H^p(s)\right]$ remains invariant over any unitary transformation.

$$I_2 = \sum_n H_{nn}^2(s) + \sum_n \sum_{m \neq n} |H_{nm}(s)|^2 = I_2^{\mathrm{diag}}(s) + I_2^{\mathrm{off}}(s)$$

$\frac{\mathrm{d}I_2^{\mathrm{diag}}(s)}{\mathrm{d}s} \geq 0$ if $\eta_{nk}(s) = H_{nk}(s)\, f\left(H_{nn}(s) - H_{kk}(s)\right)$ where $f(-x) = -f(x)$

F. Wegner: *Ann. Phys. (Leipzig)* **506**, 77 (1994)

Block diagonalization of the "Kamiltonian"

For descrete flow:

$$\mathcal{K}\left(s+1,t\right)=\mathrm{e}^{\mathrm{i}\mathcal{S}(s,t)}\mathcal{K}\left(s,t\right)\mathrm{e}^{-\mathrm{i}\mathcal{S}(s,t)}-\mathrm{i}\hbar\mathrm{e}^{\mathrm{i}\mathcal{S}(s,t)}\frac{\partial\mathrm{e}^{-\mathrm{i}\mathcal{S}(s,t)}}{\partial t}$$

For continuous flow:

$$\frac{\partial\mathcal{K}\left(s,t\right)}{\partial s}=\mathrm{i}\left[\mathcal{S}\left(s,t\right),\mathcal{K}\left(s,t\right)\right]-\hbar\frac{\partial\mathcal{S}\left(s,t\right)}{\partial t}$$

Over the flow, the Kamiltonian preserves Floquet structure

$$\mathcal{K}\left(s,t\right)=\hbar\omega N\otimes\mathbf{1}_{\mathscr{H}}+\sum_{m=-\infty}^{+\infty}P_m\otimes H^{(m)}\left(s,t\right)$$

with

$$N=\sum_{n=-\infty}^{+\infty}\left|n\right\rangle n\left\langle n\right|$$

$$P_m=\sum_{n=-\infty}^{+\infty}\left|n+m\right\rangle\left\langle n\right|$$

if $\mathcal{S}\left(s,t\right)=\sum_{m=-\infty}^{+\infty}P_m\otimes S^{(m)}\left(s,t\right)$ where $\left[S^{(m)}\left(s,t\right)\right]^{\dagger}=S^{(-m)}\left(s,t\right)$

Generator for the discrete flow in the presence of high-frequency

We assume that: $h^{(m)}(t) \sim \mathcal{O}\left((\hbar\omega)^0\right)$ $\quad \frac{\mathrm{d}^j h^{(m)}(t)}{\mathrm{d}t^j} \sim \mathcal{O}\left((\hbar\omega)^0\right)$

Each Fourier harmonic can be presented as an expansion in the powers of inverse frequencies:

$$H^{(m)}(s,t) = H_0^{(m)}(s,t) + H_1^{(m)}(s,t) + H_2^{(m)}(s,t) + \mathcal{O}\left((\hbar\omega)^{-3}\right)$$

For example, at the begining of the flow (s=0):

$$H^{(m)}(0,t) = h^{(m)}(t) + \mathbf{0}_{\mathscr{H}} + \mathbf{0}_{\mathscr{H}} + \mathcal{O}\left((\hbar\omega)^{-3}\right)$$

The main idea is, at each flow step, to eliminate the leading order term in the expansion for the non-zero Fourier harminics:

$$H^{(m\neq 0)}(s,t) = \sum_{i=s}^{+\infty} H_i^{(m)}(s,t)$$

Effective Hamiltonian $h_{\mathrm{eff}}(t) = \sum_{i=0}^{s-1} H_i^{(0)}(s,t) + \mathcal{O}\left((\hbar\omega)^{-s}\right)$

Surprisingly, such scenario can be achieved with a generator

$$\mathrm{i}\mathcal{S}(s,t) = \sum_{m\neq 0} \frac{P_m}{m} \otimes \frac{H_s^{(m)}(s,t)}{\hbar\omega}$$

V. Novičenko, G. Žlabys, E. Anisimovas: *Phys. Rev. A* **105**, 012203 (2022)

Generator for continuous flow adapted to block-diagonalize the Kamiltonian

Partial inner product $\langle n|\, \mathrm{i}\mathcal{S}\, |k\rangle$ can be interpreted as an element of a block matrix.

$$\eta_{nk}(s) = H_{nk}(s)\, f\left(H_{nn}(s) - H_{kk}(s)\right)$$

↓

$$\langle n|\, \mathrm{i}\mathcal{S}(s,t)\, |k\rangle = \langle n|\, \mathcal{K}(s,t)\, |k\rangle\, f\left(\langle n|\, \mathcal{K}(s,t)\, |n\rangle - \langle k|\, \mathcal{K}(s,t)\, |k\rangle\right)$$

where the function $f(\cdot)$ is generalized to act on the operator: $f(x\mathbf{1}_{\mathscr{H}}) = \mathbf{1}_{\mathscr{H}} f(x)$

For the case of $f(x) = x$ we obtain generator: $\mathrm{i}\mathcal{S}(s,t) = \hbar\omega \sum_{m\neq 0} m P_m \otimes H^{(m)}(s,t)$

Flow equations:

$$\frac{\partial H^{(0)}(s,t)}{\partial s} = \frac{2}{\hbar\omega} \sum_{m=1}^{+\infty} m \left[H^{(m)}(s,t)\, , H^{(-m)}(s,t)\right]$$

$$\frac{\partial H^{(n)}(s,t)}{\partial s} = -n^2 H^{(n)}(s,t) + \frac{\mathrm{i}}{\omega} n \dot{H}^{(n)}(s,t) + \frac{1}{\hbar\omega} \sum_{m\neq n} (m-n) \left[H^{(m)}(s,t)\, , H^{(n-m)}(s,t)\right]$$

A. Verdeny, A. Mielke, F. Mintert: *Phys. Rev. Lett.* **111**, 175301 (2013)

Toda block-diagonalizator which preserves band structure

In order to preserve the band structure of the Hamiltonian during the flow we can use following generator form:

$$\eta_{nk}(s) = H_{nk}(s)\,\mathrm{sgn}(n-k)$$

$$\downarrow$$

$$\mathrm{i}\mathcal{S}(s,t) = \sum_{m\neq 0} \frac{\mathrm{sgn}(m)}{\hbar\omega} P_m \otimes H^{(m)}(s,t)$$

If $H^{(m)}(0,t) = \mathbf{0}_{\mathscr{H}}$ then $H^{(m)}(s,t) = \mathbf{0}_{\mathscr{H}}$ for any s.

The expansion in powers of inverse frequency for the finite close algebra can be done automatically, if

$$h^{(n)}(t) = \sum_{l=1}^{L} c_l^{(n)}(t)\, G_l$$

and

$$[G_l, G_m] = \sum_{n=1}^{L} \gamma_{lmn} G_n$$

A. Mielke: *Eur. Phys J. B* **5**, 605 (1998)

Example of Rabi model

Initial Hamiltonian:

$$h_{\text{lab}}\left(\omega t, t\right) = \frac{\hbar\Omega}{2}\sigma_z + 2g\left(t\right)\cos\left(\omega t + \phi\right)\sigma_x$$

We assume that $\hbar\Omega - \hbar\omega = \Delta \sim \mathcal{O}\left(\left(\hbar\omega\right)^0\right)$

$$\downarrow \quad \tilde{U}\left(t\right) = \exp\left[-\mathrm{i}\omega t\sigma_z/2\right]$$

$$h^{(0)} = \frac{\Delta}{2}\sigma_z + g\left(t\right)\cos\left(\phi\right)\sigma_x + g\left(t\right)\sin\left(\phi\right)\sigma_y$$

$$h^{(2)} = \left[h^{(-2)}\right]^{\dagger} = \frac{g\left(t\right)}{2}\mathrm{e}^{\mathrm{i}\phi}\sigma_x + \frac{g\left(t\right)}{2}\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}\sigma_y$$

toda_script.nb - Wolfram Mathematica 12.1

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```
(*Assertion message definition*)
AbortAssert::trace = "Non-zero solutions for higher Fourier components present.";
AbortAssert[test_] := Check[TrueQ[test] || Message[AbortAssert::trace], Abort[]]

(* Check that at the end of the flow all non-zero Fourier components have vanished and print the results.
    The obtained coefficients h_j multiply the generators G_j.
*)
assertFlag = True;
Do[If[i[[1]][[1]] != 0, If[i[[2]] =!= 0, assertFlag = False; Break[]]], {i, resList}];
If[assertFlag == True, Print["Solutions:"]; Do[If[i[[1]][[1]] == 0, Print[TraditionalForm[h_Position[resList,i][[1,1]] == i[[2]]]]], {i, resList}], AbortAssert[assertFlag]];

calculating order:1
calculating order:2
calculating order:3
calculating order:4
calculating order:5
Simplifying...
Solutions:
```

$$h_1 = -\frac{g(t)\left(\cos(\phi)\left(6g'(t)^2 - 7g(t)g''(t) + 7\Delta^2 g(t)^2 + 8g(t)^4\right) + 4\Delta g(t)\sin(\phi)g'(t)\right)}{64\omega^4} + \frac{1}{256\omega^5}\left(\sin(\phi)\left(g'(t)\left(12g'(t)^2 - 12g(t)g''(t) + 11\Delta^2 g(t)^2 + 16g(t)^4\right) - g(t)^2 g^{(3)}(t)\right) + \Delta g(t)\cos(\phi)\left(36g'(t)^2 - 41g(t)g''(t) + 15\Delta^2 g(t)^2 + 8g(t)^4\right)\right) + \frac{g(t)^2\left(\sin(\phi)g'(t) + 3\Delta g(t)\cos(\phi)\right)}{16\omega^3} - \frac{g(t)^3\cos(\phi)}{4\omega^2} + g(t)\cos(\phi)$$

$$h_2 = \frac{g(t)\left(4\Delta g(t)\cos(\phi)g'(t) - \sin(\phi)\left(6g'(t)^2 - 7g(t)g''(t) + 7\Delta^2 g(t)^2 + 8g(t)^4\right)\right)}{64\omega^4} + \frac{1}{256\omega^5}\left(\Delta g(t)\sin(\phi)\left(36g'(t)^2 - 41g(t)g''(t) + 15\Delta^2 g(t)^2 + 8g(t)^4\right) + \cos(\phi)\left(g(t)^2 g^{(3)}(t) - g'(t)\left(12g'(t)^2 - 12g(t)g''(t) + 11\Delta^2 g(t)^2 + 16g(t)^4\right)\right)\right) + \frac{g(t)^2\left(3\Delta g(t)\sin(\phi) - \cos(\phi)g'(t)\right)}{16\omega^3} - \frac{g(t)^3\sin(\phi)}{4\omega^2} + g(t)\sin(\phi)$$

$$h_3 = \frac{\Delta}{2} + \frac{g'(t)^2 - g(t)g''(t) + 2\Delta^2 g(t)^2}{16\omega^3} + \frac{\Delta\left(-3g'(t)^2 + 3g(t)g''(t) - 2\Delta^2 g(t)^2 + g(t)^4\right)}{32\omega^4} + \frac{1}{256\omega^5}\left(4g(t)^2\left(2\Delta^4 + g'(t)^2\right) + 24\Delta^2 g'(t)^2 + g(t)\left(g^{(4)}(t) - 24\Delta^2 g''(t)\right) - 4g^{(3)}(t)g'(t) - 10g(t)^3 g''(t) + 3g''(t)^2 - 14\Delta^2 g(t)^4 - 12g(t)^6\right) - \frac{\Delta g(t)^2}{4\omega^2} + \frac{g(t)^2}{2\omega}$$

The end