Control of synchronization in complex oscillator networks via time-delayed feedback

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Motivation

The synchronous behavior can be desirable or harmful.

- Power grids
- Parkinson’s disease, essential tremor
- Pedestrians on a bridge
- Cardiac pacemaker cells
- Internal circadian clock

The ability to control synchrony in oscillatory networks covers a wide range of real-world applications.

Phase reduction method

Phase reduction method allows the approximation of high dimensional dynamics of oscillators with a single-phase variable.

$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has periodic solution $\boldsymbol{\xi}(t+T) = \boldsymbol{\xi}(t)$

⬇

$\dot{g} = 1$ phase gradually increase from 0 to T

$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + £\mathbf{g}(\mathbf{x}, t)$

⬇

$\dot{g} = 1 + £\mathbf{z}^T(g) \cdot \mathbf{g}(\boldsymbol{\xi}(g), t)$

Here $\mathbf{z}(g)$ is a phase response curve – the periodic solution of the adjoint equation $\dot{\mathbf{z}} = -[D\mathbf{f}(\boldsymbol{\xi})]^T \mathbf{z}$

Initial condition for the phase response curve: $\mathbf{z}^T(0) \cdot \dot{\boldsymbol{\xi}}(0) = 1$

$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-²)) + £\mathbf{g}(\mathbf{x}(t), t)$

⬇

$\dot{g} = 1 + £\mathbf{z}^T(g) \cdot \mathbf{g}(\boldsymbol{\xi}(g), t)$

$$\dot{\mathbf{z}}^T(t) = -\mathbf{z}^T(t)\mathbf{A}(t) - \mathbf{z}^T(t+²)\mathbf{B}(t+²)$$

where the matrices
$$\mathbf{A}(t) = D_1\mathbf{f}(\boldsymbol{\xi}(t), \boldsymbol{\xi}(t-²))$$
$$\mathbf{B}(t) = D_2\mathbf{f}(\boldsymbol{\xi}(t), \boldsymbol{\xi}(t-²))$$

Initial condition for the phase response curve: $\mathbf{z}^T(0)\dot{\boldsymbol{\xi}}(0) + \int_{-²}^{0} \mathbf{z}^T(² + s)\mathbf{B}(² + s)\dot{\boldsymbol{\xi}}(s)\,ds = 1$

V. Novičenko, K. Pyragas, *Physica D* **241**, 1090–1098 (2012)
K. Kotani et al, *Phys. Rev. Lett.* **109**, 044101 (2012)

Complex oscillator network – the phase reduction approach

Weakly coupled near-identical limit cycle oscillators:

without control

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \varepsilon \mathbf{f}_i(\mathbf{x}_i) + \varepsilon \sum_j a_{ij} \mathbf{g}(\mathbf{x}_i, \mathbf{x}_j)$$

$$\varphi_i(t)$$

$$\theta_i(t) = \Omega_i \varphi_i(t) - \Omega t \quad \text{where} \quad \Omega_i = \frac{2\pi}{T_i}$$

$$\psi_i(t) = \text{ average } \theta_i(t) \text{ over the period } T$$

$$\dot{\psi}_i = \mu_i + \varepsilon \sum_j a_{ij} h(\psi_j - \psi_i)$$

here the frequencies $\mu_i = \Omega_i - \Omega$

Synchronization condition:

$$\dot{\psi}_1 = \dot{\psi}_2 = \ldots = \dot{\psi}_N$$

under the delayed feedback control

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \varepsilon \mathbf{f}_i(\mathbf{x}_i) + \varepsilon \sum_j a_{ij} \mathbf{g}(\mathbf{x}_i, \mathbf{x}_j) + \mathbf{K}\left[\mathbf{x}_i(t - \tau_i) - \mathbf{x}_i(t)\right]$$

$$\left[\mathbf{x}_i(t - \tau_i) - \mathbf{x}_i(t)\right] \approx \left[\mathbf{x}_i(t - T_i) - \mathbf{x}_i(t)\right] + \dot{\mathbf{x}}_i(t - T_i)(T_i - \tau_i)$$

By treating a free oscillator as

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \varepsilon \mathbf{f}_i(\mathbf{x}_i) + \mathbf{K}\left[\mathbf{x}_i(t - T_i) - \mathbf{x}_i(t)\right]$$

Applying the phase reduction method for systems with time-delay

$$\dot{\psi}_i = \mu_i^{\text{eff}} + \varepsilon^{\text{eff}} \sum_j a_{ij} h(\psi_j - \psi_i)$$

$$\varepsilon^{\text{eff}} = \kappa(\mathbf{K})\varepsilon \qquad \mu_i^{\text{eff}} = \mu_i + \Omega \frac{\tau_i - T_i}{T}\left[\kappa(\mathbf{K}) - 1\right]$$

V. Novičenko, *Phys. Rev. E* **92**, 022919 (2015)

Control of synchronization in a complex oscillator network

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + £\mathbf{f}_i(\mathbf{x}_i) + £\sum_j a_{ij}\mathbf{g}(\mathbf{x}_i,\mathbf{x}_j) + \mathbf{K}\left[\mathbf{x}_i\left(t - {}^{2}{}_i\right) - \mathbf{x}_i(t)\right]$$

⬇

$$\dot{\cdot}_i = \mu_i^{\text{eff}} + £^{\text{eff}}\sum_j a_{ij}h\left(\cdot_j - \cdot_i\right)$$

Let's say $\mathbf{K} = \begin{bmatrix} K & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$£^{\text{eff}} = \sim(K)£ \qquad \mu_i^{\text{eff}} = \mu_i + \Omega\frac{{}^{2}{}_i - T_i}{T}\left[\sim(K) - 1\right] \qquad \sim(K) = \frac{1}{1+KC}$$

where $C = \int_0^T z^{(1)}(s)¶^{(1)}(s)ds$

(i) The delay times are the same $\;{}^{2}{}_i = {}^{2} = T$

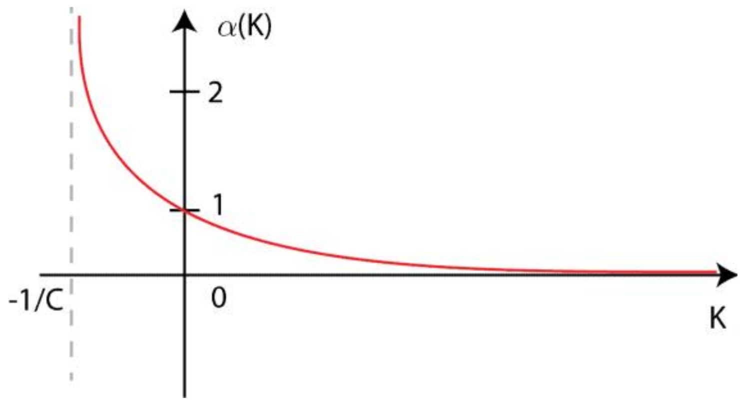
$\mu_i^{\text{eff}} = \sim(K)\mu_i \Rightarrow$ synchronization cannot be controlled

(ii) The delay times are equal to the natural periods

$${}^{2}{}_i = T_i \Rightarrow \mu_i^{\text{eff}} = \mu_i$$

(iii) The delay times are

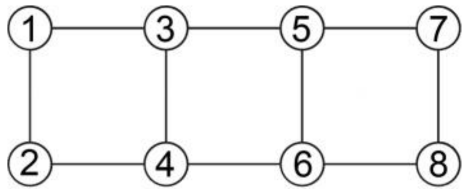
$$\frac{{}^{2}{}_i - T_i}{T} = \frac{\mu_i}{\Omega\left[1 - \sim(K)\right]} \Rightarrow \mu_i^{\text{eff}} = 0$$



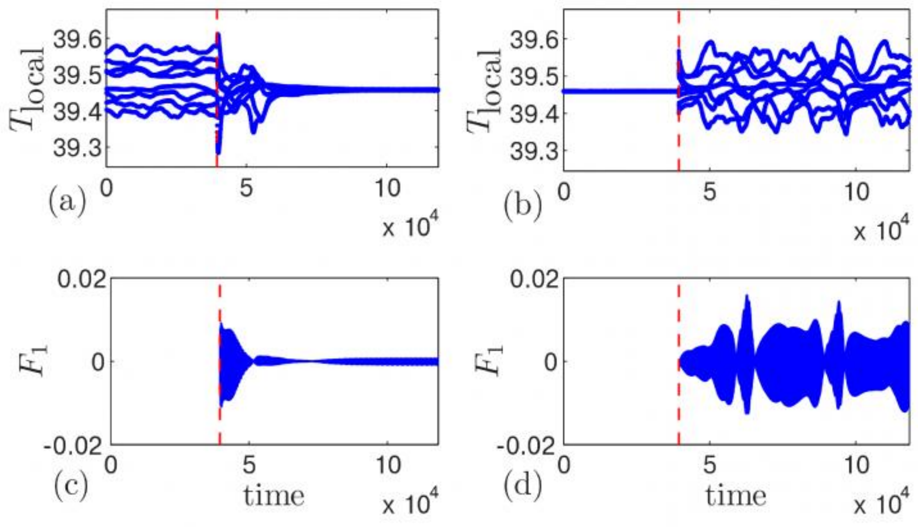
$\cdot_1(t) = \cdot_2(t) = \ldots = \cdot_N(t)$ is a stable solution, under additional assumptions: $h(0) = 0, \;\; h'(0) > 0$

V. Novičenko, *Phys. Rev. E* **92**, 022919 (2015)

8 FitzHugh-Nagumo oscillators:



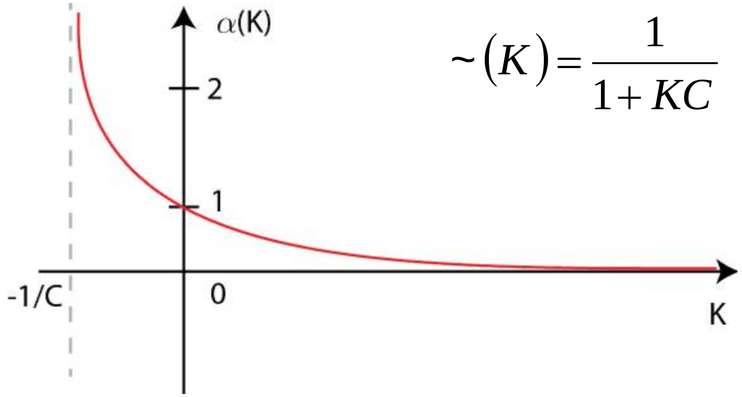
$£ = 5\times10^{-5}$ $£^{\text{eff}} = 6\times10^{-4}$ $£ = 10^{-3}$ $£^{\text{eff}} = 1.6\times10^{-4}$



$$F_1(t) = K\left[x_1(t - T_1) - x_1(t)\right]$$

V. Novičenko, *Phys. Rev. E* **92**, 022919 (2015)

Odd number limitation



$$\sim(K) = \frac{1}{1+KC}$$

$$\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i) + \mathbf{K}\left[\mathbf{x}_i(t-T_i) - \mathbf{x}_i(t)\right]$$

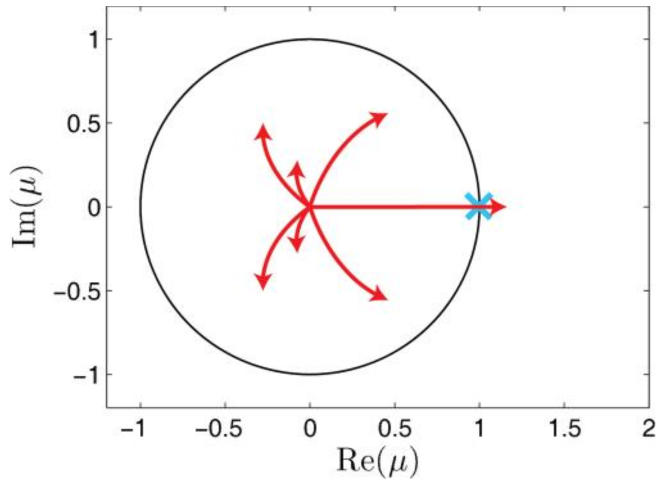
According to the odd number limitation theorem, the periodic solution $\boldsymbol{\xi}_i(t+T_i) = \boldsymbol{\xi}_i(t)$ is unstable, if

$$KC < -1$$

E. W. Hooton and A. Amann, Phys. Rev. Lett. **109**, 154101 (2012)

What happen for $K \to -1/C$?

Motion of the Floquet multipliers



Summary

The delayed feedback control force applied to a limit cycle oscillator changes its stability properties and, as a consequence, perturbation-induced phase response. The phase model of the oscillator network shows that the coupling strength and the frequencies depend on the parameters of the control.

Advantages:

- does not require any information about the oscillator model
- does not depends on network topology
- can be simple realized in experiment
- theoretically synchronization can be controlled for the arbitrary small/large coupling strength
- the control scheme has only two parameters: control gain and delay time

Disadvantages:

- the phase model can be derived only for a weak coupling
- the control force can disrupt the stability of periodic orbit

The end