# Harmonic fluid approach to Bosons and Fermions

Bernd Schmidt

Fachbereich Physik
TU Kaiserslautern

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# Outline

# Introduction

The harmonic fluid (HF) approach:

- The HF-approach allows to describe many different one-dimensional (1D) quantum systems within the same framework.
- A universality class of systems with gapless excitations can be described (approximated) by one and the same harmonic fluid Hamiltonian.
- It can be applied to fermionic as well as bosonic systems.
- The HF-Hamiltonian is quadratic in its two canonically conjugated operators. The dispersion relation (the spectrum) is linear.
- External input is needed from exactly solvable systems. High momentum cutoffs must be introduced.
- Why interesting ? : One-dimensional systems have become more and more experimentally relevant.

# History

- 1950: S. Tomonaga considers a Hamiltonian which describes a collection of harmonic oscillators, whose quanta, the ’phonons’ correspond to low-energy density and phase fluctuations.
- 1963: J. M. Luttinger finds the exact solution for an interacting fermionic system which has a linear spectrum of free Bosons.
- 1975: First application to a bosonic system by K. B. Efetov and A. I. Larkin.
- 1980: F. D. M. Haldane argues that almost all gapless 1D Fermi systems are equivalent to the Luttinger model, at least in lowest order pertubation theory.
- 1981: F. D. M. Haldane shows that a harmonic fluid description applies quite generally to 1D quantum systems of Fermions as well as Bosons.
- 2004: M. A. Cazalilla calculates correlations of interacting homogeneous Fermi- and Bose gases for different boundary conditions and finite temperature using the harmonic fluid approach. He applies it particularly to the Lieb-Liniger Model of $\delta$-interacting bosons.

# Linearization of a nonlinear Hamiltonian

Example: Spinless Bose (or Fermi) fluid:

$$H = \frac{\hbar^2}{2m} \int dx |\partial_x \Psi(x)|^2 + \frac{1}{2} \int dx \int dy V(x-y)\rho(x)\rho(y)$$

$$[\Psi(x), \hat{\Psi}^\dagger(x')] = \delta(x - x')$$

Density-Phase representation:

$$\Psi^\dagger(x) = \sqrt{\rho(x)} e^{i\varphi(x)}$$

Long wavelength approximation:

$$\rho(x) \approx \rho_0 + \Pi(x)$$

Lineralization in $\Pi(x)$ leads to the harmonic Hamiltonian (Haldane):

$$H = \frac{\hbar^2}{2\pi} \int dx [v_J(\partial_x \varphi(x))^2 + v_N(\pi \Pi(x))^2]$$

$$[\varphi(x), \Pi(x')] = i\delta(x - x')$$

# The density operator in first quantization

The density operator in first quantization

$$\rho(x) = \sum_{i=1}^{N} \delta(x - x_i)$$

can equivalently been written as

$$\rho(x) = \partial_x \Theta(x) \sum_{n=-\infty}^{+\infty} \delta(\Theta(x) - n\pi)$$

by using

$$\delta[f(x)] = \frac{\delta(x - x_0)}{|f'(x_0)|}, \text{ where } f(x_0) = 0.$$

# Long wavelength approximation

Poisson's summation formula:

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dz f(z) e^{2m\pi iz}$$

which yields:

$$\rho(x) = \partial_x \Theta(x) \sum_{n=-\infty}^{+\infty} \delta(\Theta(x) - n\pi) = \frac{1}{\pi} \partial_x \Theta(x) \sum_{m=-\infty}^{+\infty} e^{2mi\Theta(x)}$$

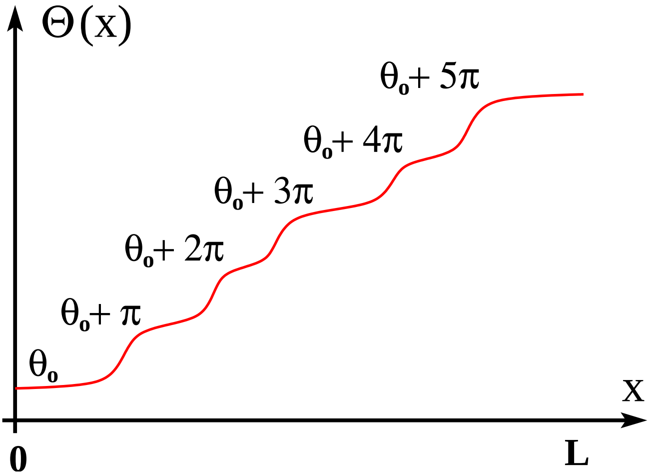
from which the long wavelength approximation ($m = 0$ term)

$$\rho(x) \approx \rho_0 + \Pi(x) = \frac{1}{\pi} \partial_x \Theta(x)$$

can be identified.
Definition:

$$\theta(x) := \Theta(x) - \rho_0 x \Rightarrow \partial_x \theta = \pi \Pi(x)$$

# Visualization of $\Theta(x)$




Picture Source: M. A. Cazalilla, J. Phys B:At. Mol. Opt. Phys. **37** (2004) S1-S47

# Periodic Boundary Conditions

Let $\Psi(x)$ obey periodic boundary conditions

$$\Psi(x+L) = \Psi(x).$$

For the particle number operator we have

$$N = \int_0^L \rho(x)dx = \int_0^L (\rho_0 + \Pi(x))dx = \int_0^L \frac{1}{\pi}\partial_x\Theta(x)dx = \frac{1}{\pi}[\Theta(L) - \Theta(0)].$$

This suggests

$$\Theta(x+L) = \Theta(x) + \pi N,$$

where $N$ is the particle number operator. Furthermore

$$\varphi(x+L) = \varphi(x) + \pi J,$$

where $J$ is an operator whose eigenvalues are <u>even integers</u>

# Open Boundary Conditions (weak)

Let the current density $j$ obey the condition

$$j(x=0)=0.$$

From the continuity equation

$$\partial_t \rho(x,t) + \partial_x j(x,t) = 0$$

and using that

$$\rho(x,t) \approx \partial_x \Theta(x)/\pi,$$

it follows that the current density

$$j(x,t) \approx -\partial_t \Theta(x,t)/\pi.$$

Demanding $j(x=0)=0$ amounts to

$$\partial_t \Theta(x=0,t) = 0.$$

# Open Boundary Conditions (strong)

Let $\rho(x)$ obey the boundary condition

$$\rho(x = 0) = 0.$$

From the expression

$$\rho(x) = \partial_x \Theta(x) \sum_{n=-\infty}^{+\infty} \delta(\Theta(x) - n\pi)$$

one can see that $\rho(x = 0) = 0$ provided that $\Theta(x = 0) \neq n\pi$, where $n$ is an integer. The property

$$[\Theta(L) - \Theta(0)] = N\pi$$

also fixes $\Theta(L) = \Theta(0) + N\pi$.

# Diagonalization of the HF-Hamiltonian

Harmonic Fluid Hamiltonian:

$$H = \frac{\hbar^2}{2\pi} \int dx [v_J (\partial_x \varphi(x))^2 + v_N (\partial_x \theta(x))^2]$$

Mode Expansions (PBC):

$$\Theta(x) = \theta_0 + \frac{\pi x}{L} N + \frac{1}{2} \sum_{q \neq 0} \left| \frac{2\pi K}{qL} \right|^{1/2} e^{-a|q|/2} [e^{iqx} b(q) + e^{-iqx} b^\dagger(q)]$$

$$\varphi(x) = \theta_0 + \frac{\pi x}{L} J + \frac{1}{2} \sum_{q \neq 0} \left| \frac{2\pi K}{qL} \right|^{1/2} e^{-a|q|/2} \, \mathrm{sgn}(q) [e^{iqx} b(q) + e^{-iqx} b^\dagger(q)]$$

$K = \sqrt{v_J / v_N}$, $v_s = \sqrt{v_N v_J}$, $a$ : momentum cutoff

Result:

$$H = \sum_{q \neq 0} \hbar v_s |q| b^\dagger(q) b(q) + \frac{\hbar \pi}{2L} [v_N (N - N_0)^2 + v_J J^2] + \text{const}$$

# Momentum cutoff

For short-range interactions,

$$a^{-1} \lesssim q_c.$$

$q_c$ is then fixed by demanding that

$$\hbar v_s q_c = \mu$$

It is an estimate of the momentum where the excitation spectrum deviates from the linear behavior.

# Field Operator

$$\Psi^\dagger(x) = \sqrt{\rho(x)} e^{i\phi(x)}$$

$\rightarrow$ Calculate the square root of the density operator!

$$\rho(x) = [\rho_0 + \Pi(x)] \sum_{n=-\infty}^{+\infty} \delta(\Theta(x) - n\pi).$$

Fermi's trick:

$$[\delta(y)]^2 = A\delta(y) \Rightarrow \sqrt{\delta(y)} = A^{-1/2}\delta(y)$$

where the constant A depends on the particular way the Dirac delta function is defined.

Using Poisson's formula again:

$$\Psi^\dagger(x) \propto \sqrt{\rho_0 + \Pi(x)} \sum_{m=-\infty}^{+\infty} e^{2mi\Theta(x)} e^{i\phi(x)}$$

Construction of the Fermi operator by the Jordan Wigner type transformation

$$\Psi_F^\dagger(x) = \Psi^\dagger(x) e^{i\Theta(x)}$$

# Correlations at $T = 0$ (PBC)

Bosons:

$$\langle \Psi^\dagger(x)\Psi(0)\rangle =$$

$$\rho_0 \left\{ b_0 \left[\frac{1}{\rho_0 d(x|L)}\right]^{1/2K} + \sum_{m=1}^{+\infty} b_m \left[\frac{\pi}{\rho_0 d(x|L)}\right]^{2m^2K+1/2K} \cos(2\pi m \rho_0 x) \right\}$$

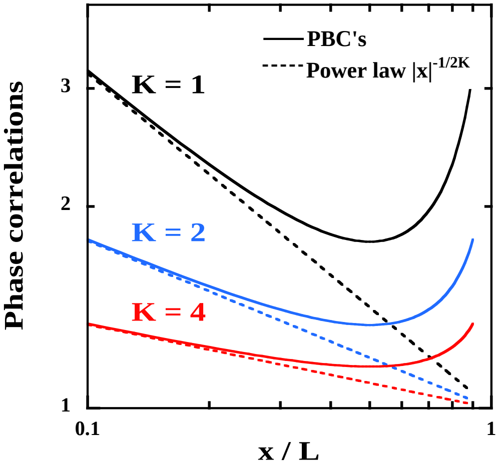
Fermions:

$$\langle \Psi_F^\dagger(x)\Psi_F(0)\rangle =$$

$$\rho_0 \sum_{m=0}^{+\infty} b_m \left[\frac{\pi}{\rho_0 d(x|L)}\right]^{2(m+\frac{1}{2})^2K+1/2K} \sin\left|2\pi\left(m+\frac{1}{2}\right)\rho_0 x\right|$$

$$d(x|L) = L\sin(\pi x/L)/\pi \xrightarrow{L\to\infty} x$$

# Correlations at T=0 (PBC)

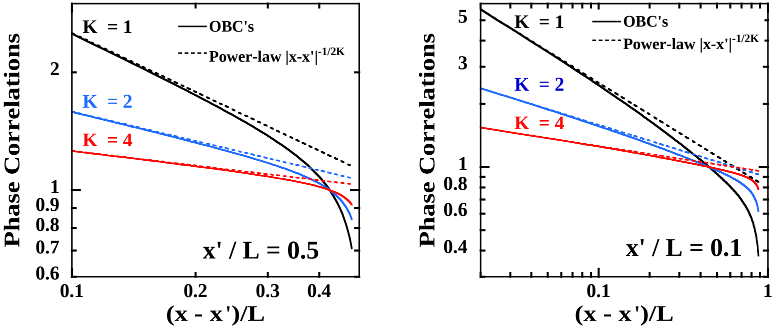



Picture Source: M. A. Cazalilla, J. Phys B:At. Mol. Opt. Phys. **37** (2004) S1-S47

# Correlations at T=0 (open boundary cond.)

$$\langle \Psi^\dagger(x)\Psi(0)\rangle = \rho_0 \left[ \frac{\rho_0^{-1}\sqrt{d(2x|2L)d(2x'|2L)}}{d(x+x'|2L)d(x-x'|2L)} \right]^{1/2K} + \ldots$$

$$d(x|L) = L\sin(\pi x/L)/\pi \xrightarrow{L\to\infty} x$$

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# Correlations at finite temperature

Bosons:

$$\langle \Psi(x)\Psi(0)\rangle_T = \rho_0 C \left[\frac{\pi/L_T}{\rho_0 \sinh(\pi x/L_T)}\right]^{\frac{1}{2K}} + \ldots$$

Fermions:

$$\langle \Psi_F(x)\Psi_F(0)\rangle_T = \rho_0 A \left[\frac{\pi/L_T}{\rho_0 \sinh(\pi x/L_T)}\right]^{\frac{K}{2}+\frac{1}{2K}} \sin|\pi\rho_0 x| + \ldots$$

Thermal length:

$$L_T = \hbar v_s/T$$

# How to calculate $v_J$ and $v_N$

- Hamiltonian of the system is

$$H = \sum_{q \neq 0} \hbar v_s |q| b^\dagger(q) b(q) + \frac{\hbar\pi}{2L}\left[v_N (N - N_0)^2 + v_J J^2\right] + \text{const}$$

- So we have

$$v_N = \frac{L}{\pi\hbar}\left[\frac{\partial^2 E(N)}{\partial N^2}\right]_{N=N_0} = \frac{1}{\pi\hbar}\left(\frac{\partial\mu}{\partial\rho}\right)$$

- And following from the derivation of the harmonic fluid Hamiltonian, we have

$$v_J = \hbar\pi\rho_0/M$$

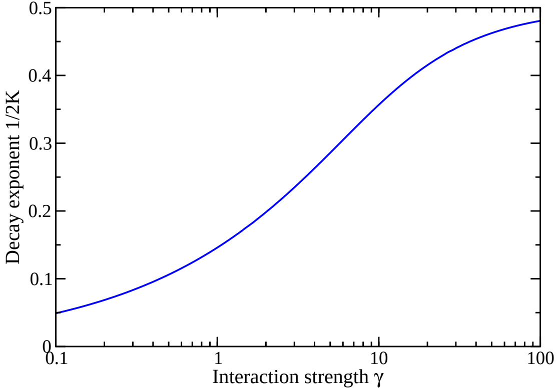
- $\frac{\partial\mu}{\partial\rho}$ must be taken from exact analytic solutions of the system in question, for example from the Bethe ansatz solution of the delta-interacting Bosons.

# Decay exponent of a 1D-Bose gas . . .

. . . with $\delta$-interaction and T=0.
From harmonic fluid approach:

$$\langle \Psi^\dagger(x)\Psi(0)\rangle \propto x^{-\frac{1}{2K}}$$

$$K = \sqrt{v_J/v_N} = \pi\hbar\sqrt{\frac{\rho}{m}}\left(\frac{\partial\mu}{\partial\rho}\right)^{-1/2}$$



# Path integral formulation of HF

Advantage: A more rigorous treatment is possible.

Partition function of a bosonic system in the grand canonical ensemble as a coherent-state path integral

$$Z = \int [d\psi^* d\psi] e^{-S[\psi^*,\psi]}$$

The functional $S[\psi^*, \psi]$ is the Euclidean action.

$$S[\psi^*, \psi] = \int_0^{\hbar\beta} \frac{d\tau}{\hbar} \int_0^L dx \Big[ \hbar\psi^*(x,\tau)\partial_\tau\psi(x\tau) - \mu\psi^*(x,\tau)\psi(x,\tau)$$

$$+\frac{\hbar^2}{2m}|\partial_x\psi(x,\tau)|^2 + \frac{1}{2}\int_0^L dx' v(x-x')\psi^*(x,\tau)\psi^*(x',\tau)\psi(x',\tau)\psi^*(x,\tau)\Big]$$

# Path Integral formulation of HF

Using polar decompostion $\psi(x,\tau) = \sqrt{\rho(x,\tau)}e^{i\varphi(x,\tau)}$, where $\rho(x,\tau)$ and $\varphi(x,\tau)$ are *real functions* (i. e. not operators), the action becomes

$$S[\psi^*,\psi] = \int_0^{\hbar\beta}\frac{d\tau}{\hbar}\int_0^L dx\Big[i\hbar\rho(x,\tau)\partial_\tau\varphi(x\tau) + \frac{\hbar^2}{2m}\rho(x,\tau)(\partial_x\varphi(x,\tau))^2 + \frac{\hbar}{2}\partial_\tau\rho(x,\tau) + \frac{\hbar^2}{8m}\frac{(\partial_x\rho(x,\tau))^2}{\rho(x,\tau)} - \mu\rho(x,\tau) + \frac{1}{2}\int_0^L dx'\rho(x,\tau)v(x-x')\rho(x',\tau)\Big]$$

# Intergrating out the fast modes

First we split

$$\rho(x,\tau) = \rho_<(x,\tau) + \rho_>(x,\tau)$$
$$\varphi(x,\tau) = \varphi_<(x,\tau) + \varphi_>(x,\tau)$$

where $\rho_>(x,\tau)$ and $\varphi_>(x,\tau)$ describe the fast modes.
Definition of the effective-low energy action by

$$e^{-S_{\text{eff}}[\Theta,\varphi]} = \int [d\rho_> d\varphi_>] e^{-S[\rho_>,\varphi_>,\Theta,\varphi]}$$

In general the integral cannot be performed, but it can be justified by physical considerations and pertubation theory that the result is

$$S_{\text{eff}}[\Theta,\varphi] = \int_0^{\hbar\beta} d\tau \int_0^L dx \left[ \frac{i}{\pi}\partial_x\Theta(x,\tau)\partial_\tau\varphi(x,\tau) + \frac{\hbar}{2\pi}v_J(\partial_x\phi(x,\tau))^2 + \frac{\hbar}{2\pi}v_N(\partial_x\Theta(x,\tau) - \pi\rho_0)^2 \right]$$

# Constructive bosonization of fermionic systems

- Assume for simplicity, that the Fermi-momentum $k_F = 0$
- Assume that the spectrum has been linearized arround $k_F$
- The vacuum state $|0\rangle_0$ of the system is the "Dirac sea" of an infinite number of fermions occupying all states with $k \in (-\infty, 0]$
- Fermionic operators have the property

$$c_k|0\rangle_0 = 0,\ k > 0$$
$$c_k^\dagger|0\rangle_0 = 0,\ k \leq 0$$

- Bosonic operators are introduced by

$$b_q = \sqrt{\frac{2\pi}{Lq}} \sum_k c_{k-q}^\dagger c_k,\ (q > 0)$$
$$b_q^\dagger = \sqrt{\frac{2\pi}{Lq}} \sum_k c_{k+q}^\dagger c_k,\ (q > 0)$$

# Summary

- The harmonic fluid approach is a general framework for calculating properties of gapless 1D interacting quantum systems.
- The HF approach shows, that many 1D quantum systems are equivalent to a Hamiltonian which is quadratic in its two canonically conjugated operators.
- It works for Fermions and Bosons.
- It is valid for all regimes of interaction strength and finite temperatures.
- Density, first and second order correlations can be calculated analytically.
- The HF-approach shows that correlations in 1D at $T = 0$ decay with a power law and exponential at T>0 .
- Parameters must be determined by external input.