Quantum Zeno and anti-Zeno effects: Quantum trajectory method

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September 3, 2006

Outline

1. What is quantum Zeno effect
2. Quantum Monte-Carlo method
3. Numerical simulations of quantum Zeno effect

Why quantum Zeno effect?

Why quantum Zeno effect is interesting:

- Quantum Zeno effect is a direct consequence of the fundamental features of quantum mechanics
- Quantum Zeno effect is important for some error-correcting codes in quantum computing.

Postulates of quantum mechanics

In quantum mechanics there are two kinds of dynamical rules:

1. Unitary evolution. Evolution of the closed quantum system is determined by the Schrödinger equation
$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi\rangle = \hat{H}|\Psi\rangle.$$
2. A projective measurement is described by a Hermitian operator $\hat{M}$. The possible outcomes of the measurement correspond to the eigenvalues m of the operator. Upon measuring the state $|\Psi\rangle$, the probability of getting result m is given by
$$p(m) = \langle\Psi|\hat{P}_m|\Psi\rangle,$$
where $\hat{P}_m$ is the projector onto the eigenspace of $\hat{M}$ with eigenvalue m. Given that outcome m occurred, the state of the quantum system immediately after the measurement is
$$|\Psi\rangle \rightarrow \frac{\hat{P}_m|\Psi\rangle}{\sqrt{p(m)}}.$$

Non-exponential decay

Non-exponential decay

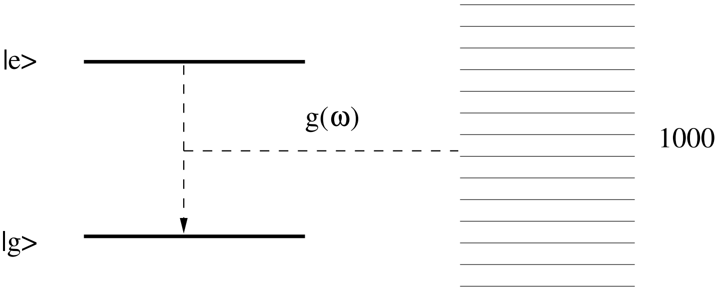
From the unitarity of the quantum evolution follows that for short durations the decay law cannot be exponential.

The proability to find the system in the initial state can be at most quadratic

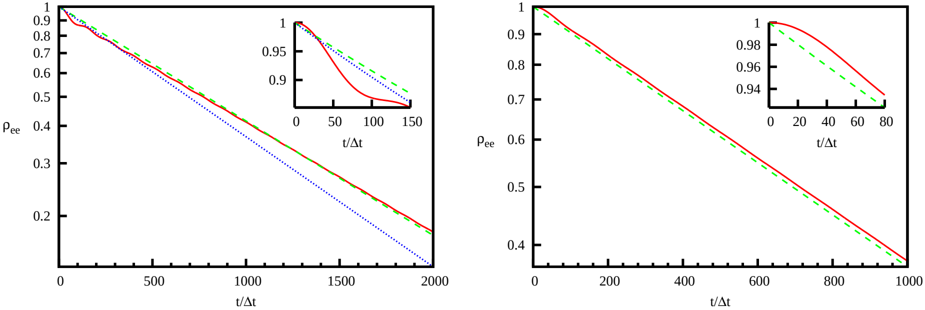
$$P(t) \approx 1 - at^2.$$

Non-exponential decay

Example: Non-exponential decay in two-level system interacting with the reservoir.



Non-exponential decay



Time dependence of the occupation of the exited level of the decaying system.

Quantum Zeno effect

Quantum Zeno effect is a consequence of two peculiarities of quantum mechanics:

1. Unitary evolution: non-exponential decay for short times, $P(t) \approx 1 - at^2$.
2. Projection postulate: after the measurement the state of the system becomes one of the eigenstates of the measured observable.

The probability to find the system in the initial state after N measurements is

$$P(T) = \left(1 - a(T/N)^2\right)^N \to 1.$$

Evolution of the frequently measured quantum system is frozen — quantum Zeno effect.

Quantum Zeno effect

Jump rate during the measurement

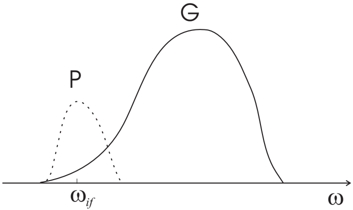
$$R(i \to f) = \frac{2\pi}{\hbar} \int_{-\infty}^{+\infty} G(\omega) P_{if}(\omega) \mathrm{d}\omega \qquad (1)$$

where

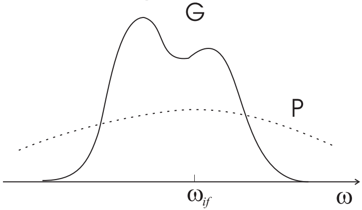
$$G(\omega) = \rho(\omega) |V_{if}(\omega)|^2,$$

$$P_{if}(\omega) = \frac{1}{\pi} \mathrm{Re} \int_0^{\infty} F_{if}(t) \mathrm{e}^{\mathrm{i}(\omega+\omega_{if})t} \, \mathrm{d}t.$$

Quantum Zeno and anti-Zeno effects



Conditions for the quantum anti-Zeno effect.



Conditions for the quantum Zeno effect.

Classical Monte-Carlo method

Let us consider the following equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\lambda(x - a). \qquad (2)$$

After short time interval Δt the variable x is

$$x(t + \Delta t) = (1 - \lambda\Delta t)x(t) + \lambda\Delta t\, a\,.$$

During the time interval Δt two possibilities can occur. Either x is equal to a with probability

$$p = \lambda\Delta t,$$

or x does not change with probability $1 - p$.

Classical Monte-Carlo method

Equation (2) can be replaced by the stochastic process:

1. Generate a random number r_n distributed uniformly on the interval $[0, 1]$.
2. Compare p with r_n and calculate $x(t_{n+1})$ according to the rule

$$x(t_{n+1}) = a, \quad p < r_n,$$
$$x(t_{n+1}) = x(t_n), \quad p > r_n.$$

Quantum Monte-Carlo method

Let us consider the system described by a master equation

$$\frac{\partial}{\partial t}\hat{\rho}(t) = \mathcal{M}\hat{\rho}. \tag{3}$$

The superoperator $\mathcal{M}$ can be separated into two parts

$$\mathcal{M} = \mathcal{L} + \mathcal{J}.$$

Since equation should preserve the trace of the density matrix, we have the equality

$$\mathrm{Tr}\{\mathcal{L}\hat{\rho}(t)\} + \mathrm{Tr}\{\mathcal{J}\hat{\rho}(t)\} = 0.$$

Quantum Monte-Carlo method

Equation (3) can be rewritten in the form

$$\hat{\rho}(t+\Delta t) = \frac{\hat{\rho}(t) + \mathcal{L}\hat{\rho}(t)\Delta t}{1 + \mathrm{Tr}\{\mathcal{L}\hat{\rho}(t)\}\Delta t}\left(1 - \mathrm{Tr}\{\mathcal{J}\hat{\rho}(t)\}\Delta t\right) + \frac{\mathcal{J}\hat{\rho}(t)}{\mathrm{Tr}\{\mathcal{J}\hat{\rho}(t)\}}\mathrm{Tr}\{\mathcal{J}\hat{\rho}(t)\}\Delta t.$$

This equation can be interpreted in the following way: during the time interval Δt two possibilities can occur.

Quantum Monte-Carlo method

Either after time Δt the density matrix is equal to conditional density matrix

$$\hat{\rho}_{\mathrm{jump}}(t+\Delta t) = \frac{\mathcal{J}\hat{\rho}(t)}{\mathrm{Tr}\{\mathcal{J}\hat{\rho}(t)\}}$$

with the probability

$$p_{\mathrm{jump}}(t) = \mathrm{Tr}\{\mathcal{J}\hat{\rho}(t)\}\Delta t$$

or to the density matrix

$$\hat{\rho}_{\mathrm{no-jump}}(t+\Delta t) = \frac{\hat{\rho}(t) + \mathcal{L}\Delta t\hat{\rho}(t)}{1 + \mathrm{Tr}\{\mathcal{L}\hat{\rho}(t)\}\Delta t}$$

with the probability $1 - p_{\mathrm{jump}}(t)$. Thus the equation (3) can be replaced by the stochastic process.

Quantum Monte-Carlo method

Further we assume that the superoperators $\mathcal{L}$ and $\mathcal{J}$ have the form

$$\mathcal{L}\hat{\rho} = \frac{1}{i\hbar}(\hat{H}_{\mathrm{eff}}\hat{\rho} - \hat{\rho}\hat{H}_{\mathrm{eff}}^{\dagger}),$$
$$\mathcal{J}\hat{\rho} = \hat{C}\hat{\rho}\hat{C}^{\dagger}.$$

The operators $\hat{H}_{\mathrm{eff}}$ and $\hat{C}$ are non-Hermitian in general.
If the density matrix at the time t factorizes as $\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|$ then after time interval Δt the density matrices $\hat{\rho}_{\mathrm{jump}}(t+\Delta t)$ and $\hat{\rho}_{\mathrm{no-jump}}(t+\Delta t)$ factorize also.
Therefore, equation for density matrix can be replaced by the corresponding equation for the state vectors.

Quantum Monte-Carlo method

The probability of a jump occurring in the time interval Δt is

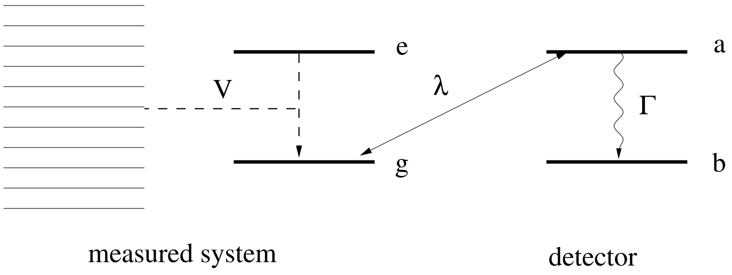
$$p_{\text{jump}}(t) = \langle\Psi(t)|\hat{C}^\dagger\hat{C}|\Psi(t)\rangle\Delta t.$$

When the wavefunction $|\Psi(t_n)\rangle$ is given, the wavefunction $|\Psi(t_{n+1})\rangle$ is determined by the following algorithm:

1. evaluate the collapse probability $p_{\text{jump}}(t_n)$
2. generate a random number r_n distributed uniformly on the interval $[0,1]$
3. compare $p_{\text{jump}}(t_n)$ with r_n and calculate $|\Psi_c(t_{n+1})\rangle$ according to the rule

$$|\Psi(t_{n+1})\rangle \sim \hat{C}|\Psi_c(t_n)\rangle, \quad p_{\text{jump}}(t_n) < r_n,$$
$$|\Psi(t_{n+1})\rangle \sim \exp\left(-\frac{i}{\hbar}\hat{H}_{\text{eff}}\Delta t\right)|\Psi(t_n)\rangle, \quad p_{\text{jump}}(t_n) > r_n.$$

Model of the measurement



The detector

The Hamiltonian of the detecting atom is

$$\hat{H}_D = \frac{\hbar\Omega_D}{2}\hat{\sigma}_z .$$

The detecting atom interacts with the electromagnetic field. The interaction of the atom with the field is described by the term

$$\mathcal{L}_D\hat{\rho}_D = -\frac{\Gamma}{2}(\hat{\sigma}_+\hat{\sigma}_-\hat{\rho}_D - 2\hat{\sigma}_-\hat{\rho}_D\hat{\sigma}_+ + \hat{\rho}_D\hat{\sigma}_+\hat{\sigma}_-).$$

Interaction with the measured system:

$$\hat{H}_I = \hbar\lambda|g\rangle\langle g|(\hat{\sigma}_+ + \hat{\sigma}_-).$$

Duration of the measurement

Two level system can act as an effective detector when the decay rate Γ is large. Then

$$F_{e,g}(t) = \rho_{bb}(t) \approx \exp\left(-\frac{t}{\tau_M}\right)$$

where

$$\tau_M = \frac{\Gamma}{2\lambda^2}$$

is the characteristic duration of the measurement.

Stochastic simmulation of the detector

The Hamiltonian of the measured system is

$$\hat{H}_A = \hbar\omega_A |e\rangle\langle e|,$$

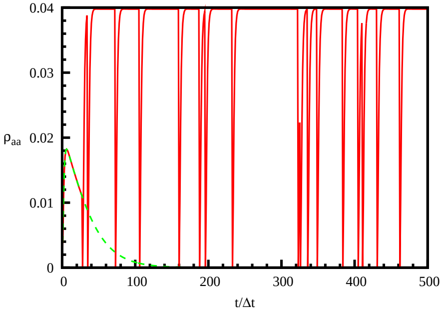
The operator $\hat{C}$ describing jumps is

$$\hat{C} = \sqrt{\Gamma}\hat{\sigma}_-,$$

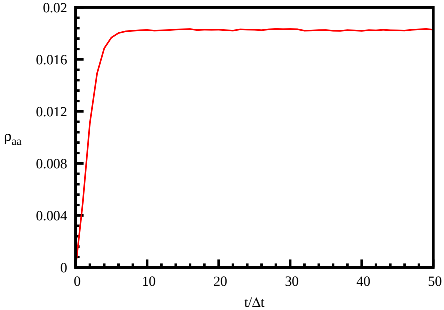
and the effective Hamiltonian

$$\hat{H}_{\mathrm{eff}} = \hat{H}_A + \hat{H}_D + \hat{H}_I - \mathrm{i}\hbar\frac{\Gamma}{2}\hat{\sigma}_+\hat{\sigma}_-.$$

Stochastic simmulation of the detector



Typical quantum trajectories of the detector.



Probability for the detector to be in the excited state, after performing an ensemble average.

Rabi oscillations

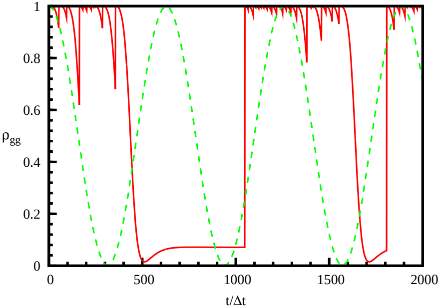
The perturbation operator is

$$\hat{V} = -\hbar\Omega_R(|e\rangle\langle g| + |g\rangle\langle e|)\cos\Omega t,$$

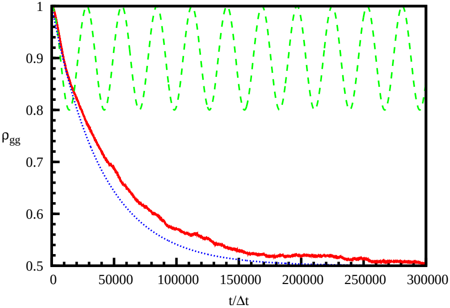
One can estimate the transition rates in the measured system as

$$\Gamma_{e\to g} \approx \Gamma_{g\to e} \approx \frac{\Omega_R^2}{2}\frac{\tau_M}{1+(\tau_M\Delta\omega)^2},$$

where $\Delta\omega = \omega_A - \Omega$ is the detuning.



Typical quantum trajectories of the measured two-level system.



Probability for the atom with the nonzero detuning to be in the ground state.

Decaying system

Fermi's golden rule:

$$\Gamma^{(0)}_{e\to g} = 2\pi\rho(\omega_A)|g(\omega_A)|^2.$$

In the numerical calculations we take the frequencies of the reservoir ω distributed in the region $[\omega_A - \Lambda, \omega_A + \Lambda]$ with the constant spacing $\Delta\omega$. Simplest choice of the interaction strength $g(\omega)$:

$$g(\omega) = g_0 \left(1 + \frac{a}{\Lambda}(\omega - \omega_A)\right).$$

The decay rate, obtained using the Laplace transform method

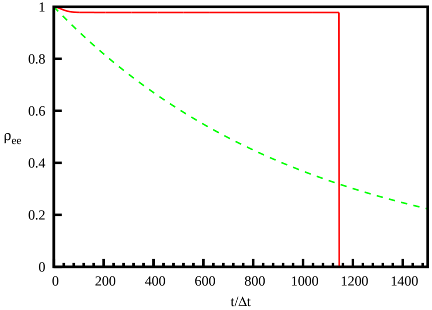
$$\Gamma^{(1)}_{e\to g} \approx \Gamma^{(0)}_{e\to g} \left(1 - \frac{\Gamma^{(0)}_{e\to g}}{\pi\Lambda}(5a^2 - 1)\right).$$

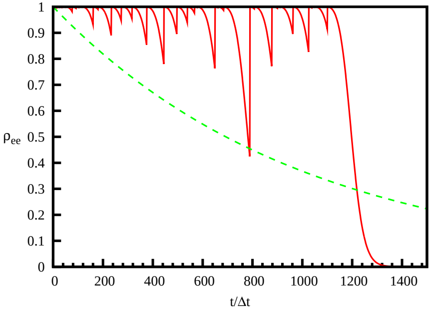
Measurement of the decaying system

- When the parameter a is zero, the decay rate of the measured system is
$$\Gamma_{e\to g} = \Gamma^{(0)}_{e\to g}\left(1 - \frac{2}{\pi}\frac{1}{\Lambda\tau_M} + \cdots\right).$$
- When a is not zero: in order to estimate the decay rate, we solve the Liouville-von Neumann equation for the density matrix of the system, including additional terms describing decay of the non-diagonal elements with rate $1/\tau_M$. The measurement-modified decay rate is

$$\Gamma_{e\to g} = \Gamma^{(0)}_{e\to g}\left(1 - \frac{\Gamma^{(0)}_{e\to g}}{\pi\Lambda}(5a^2 - 1)\right) + \Gamma^{(0)}_{e\to g}\frac{2}{\pi}\frac{(a^2-1)}{\Lambda\tau_M}.$$

Measurement of the decaying system



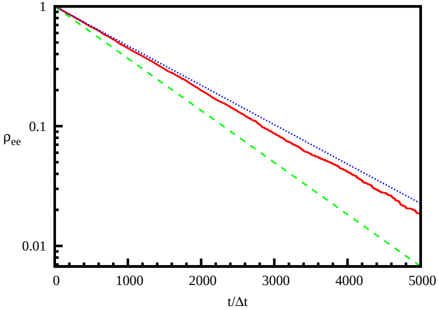


$$\hat{H}_I = \hbar\lambda|g\rangle\langle g|(\hat{\sigma}_+ + \hat{\sigma}_-).$$

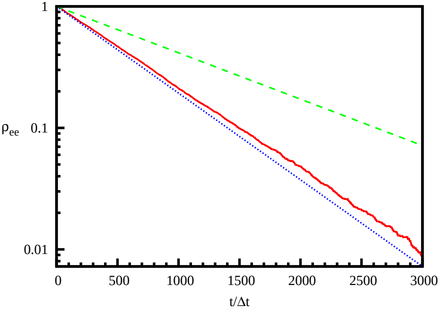
$$\hat{H}_I = \hbar\lambda|e\rangle\langle e|(\hat{\sigma}_+ + \hat{\sigma}_-).$$

Typical quantum trajectories of the measured decaying system.

Measurement of the decaying system



Quantum Zeno effect.



Quantum anti-Zeno effect.

Summary

- The quantum trajectories produced by stochastic simulations show the probabilistic behavior exhibiting the collapse of the wave-packet in the measured system, although the quantum jumps were performed only in the detector.
- The general expression (1) for the jump rate during the measurement gives good agreement with the numerical data, unless the interaction of the measured system with the reservoir is strongly mode dependent.
- The decay rates mostly depend only on one parameter — the duration of the measurement. Other details of the detector are not important.

Thank you!