Lithuanian Journal of Physics, Vol. **47**, No. 3, pp. 267–271 (2007) doi:10.3952/lithjphys.47306

DIPOLE FIELD REPRESENTATION IN THE BASIS SET OF HERMITE–GAUSSIAN FUNCTIONS *

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Received 8 June 2007

The dipole non-harmonic electromagnetic fields are derived from the Hertzian vectors expressed using an orthonormal basis set of the finite-energy Hermite–Gaussian functions. Upon application of the Fourier transform operation's eigendecomposition, four sets of the expansion coefficients are introduced. The time and space-limited function describing the dipole excitation as well as its Fourier transform are then expressed through the appropriate couples of introduced functions. The numerical calculation of dipole-radiated fields produced by a given Hertzian vector are presented and their properties are discussed.

Keywords: antennas, electromagnetic fields, signal transmission

PACS: 84.40.Ba, 84.40.Ua, 03.50.De

1. Introduction

The propagation of ultra-wideband electromagnetic signals can be formulated in terms of a generalized time-scale analysis. The analysis is natural to electromagnetic fields for the following fundamental reason: Maxwell's equations in free space are invariant under a group of transformations called the conformal group C, which includes space-time translations, dilatations, and rotations. In fact, these transformations taken together essentially characterize free electromagnetic waves. The structure of these equations also makes it possible to extend their solutions analytically to a certain domain in complex space-time, where conformal transformations act as unitary operators [1].

As long as the boundaries and material media are symmetric, the signal sources and fields may be decomposed into constituents that individually imitate the symmetry of the environment. Usually a vector field is decomposed into a lamellar component having zero curl and solenoidal component having zero divergence (known as Helmholtz decomposition). The Hertzian potentials emerge as the consequence of symmetry application to electromagnetic fields and sources [2].

The introduction of potential functions replaces the solution process dealing with two coupled first order Maxwell's equations by the one dealing with two uncoupled second order equations.

Recently we reported [3] that the wave equation solution representing Hertzian dipole fields can be expressed by a couple of different functions that are expanded in terms of basis set of Hermite–Gaussian functions. This approach allows for a unified form to represent the dipole electromagnetic fields of different origin, whether stationary or variable. The sum of the two introduced functions describes behaviour of time and space-limited electromagnetic pulses, while the difference of the functions reflects the dipole field, formally being its Fourier transform.

The purpose of this work is to define the eigendecomposition sets of the Fourier transform operation, then, using its elements and their arrays, to define a short impulse excitation function, and to compute the electric and magnetic fields radiated by an excited point dipole source.

2. Theoretical basis

2.1. Hertzian potentials

The Hertzian potentials are introduced when we choose the Lorentz gauge to remove the arbitrariness of the divergence of the vector potential. Thus we should be able to write both the electric and magnetic fields in terms of a single potential function.

* The report presented at the 37th Lithuanian National Physics Conference, 11–13 June 2007, Vilnius, Lithuania.

ISSN 1648-8504

A choice of the Lorentz gauge condition allows the vector and scalar potentials to act as finite-velocity waves. Actually it establishes a relationship between vector and scalar potentials, thus it is possible to write both the electric and magnetic fields in terms of a single potential function [2]. Usually the Hertzian potentials with electric $\vec{\pi}_\mathrm{e}$ and magnetic $\vec{\pi}_\mathrm{m}$ vectors are applied.

These potentials already incorporate the physical laws of wave propagation, and they can be used to construct arbitrary electromagnetic waves by superposition. Since the physical laws are now built into the potentials, we can concentrate on choosing the coefficient function for a given field.

2.2. Dipole field representation

We aim to compute the electric and magnetic fields radiated by a point dipole source having any temporal variation of the polarization current. Since electromagnetic fields are completely described using either $\vec{\pi}_\mathrm{e}$ or $\vec{\pi}_\mathrm{m}$, we apply an electric Hertzian potential, which at the distance r from the source located at $r = 0$ can be expressed as

$$\vec{\pi}_\mathrm{e} = \vec{p}\,\frac{f(\beta)}{4\pi r}\,, \tag{1}$$

where the function $f(\beta)$ defines temporal and spatial potential's dependence; $\beta = kct - kr$, $\vec{p}$ is the dipole moment vector, k is the propagation coefficient (in the case of harmonic field, $k = \sqrt{k_x^2 + k_y^2 + k_z^2} = 2\pi/\lambda$ is the wave number, λ is the wavelength), $c = 1/\sqrt{\varepsilon_0\mu_0}$ is the phase velocity of electromagnetic field in vacuum.

The electromagnetic field of this dipole is defined by the formulae

$$\vec{E} = \frac{k^3}{4\pi\varepsilon_0}\left\{ \big[[\vec{p}\vec{r}_0]\,\vec{r}_0\big]\,\frac{1}{kr}\,\frac{\partial^2 f(\beta)}{\partial\beta^2} + \right.$$
$$\left. + \big[3\vec{r}_0(\vec{r}_0\vec{p}) - \vec{p}\big]\left[\frac{f(\beta)}{(kr)^3} + \frac{1}{(kr)^2}\,\frac{\partial f(\beta)}{\partial\beta}\right]\right\} -$$
$$- \frac{1}{\varepsilon_0}\vec{p}\,f(\beta_0)\,\delta(r)\,, \tag{2}$$

$$\vec{B} = \frac{\mu_0 k^3}{4\pi\sqrt{\varepsilon_0\mu_0}}\,[\vec{p}\vec{r}_0]\left[\frac{1}{kr}\,\frac{\partial^2 f(\beta)}{\partial\beta^2} + \frac{1}{(kr)^2}\,\frac{\partial f(\beta)}{\partial\beta}\right], \tag{3}$$

which, in the case of harmonic excitation, coincide with well-known dipole field's expressions [4]. We denote here $\beta_0 = kct$, $\vec{r}_0 = \vec{r}/r$. The last term in (2) reflects the inner electric dipole field at $r = 0$ like in the case of a static dipole [4].

For physically realizable pulsed signals we need a basis set of functions with finite norms and energies. Functions of this kind are the Hermite–Gaussian functions which constitute an orthonormal basis.

2.3. Finite-energy signal expansion

At $r \neq 0$ the Hertzian vector (1) satisfies the wave equation for any function with requisite order derivatives. Because the wave equation's operators are all linear and hence the wave equation obeys the principle of superposition, the linear set of functions also will form the solution. The Hermite–Gaussian functions [5]

$$\phi_n(\beta) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\,\mathrm{e}^{-\beta^2/2}\,H_n(\beta) \tag{4}$$

constitute an orthonormal basis for the set of finite-energy functions. Thus any finite-energy dependence of dipole moment can be expanded in the form

$$f(\beta) = \sum_{n=0}^{\infty} a_n\,\phi_n(\beta)\,, \tag{5}$$

where expansion coefficients are

$$a_n = \int_{-\infty}^{\infty} f(\beta)\,\phi_n(\beta)\,\mathrm{d}\beta\,. \tag{6}$$

The coefficients constitute the representation of the dipole field expressed through the function $f(\beta)$ in the Hermite–Gaussian basis set. The major part of energy represented by the nth order Hermite–Gaussian function is concentrated in the arguments interval between $-(2n+1)^{1/2}$ and $(2n+1)^{1/2}$.

It is well known [6] that the Hermite–Gaussian functions $\phi_n(\beta)$ are eigenfunctions of the fractional Fourier transform operation with eigenvalues $\lambda_n = \exp(\mathrm{i}n\theta)$. It means that the fractional Fourier transform of a_n is a simple replacement of a_n by $\mathrm{e}^{\mathrm{i}n\theta}\cdot a_n$, where θ varies between 0 and 2π. The act of obtaining these coefficients is a transformation from continuous to discrete basis. Thus for any function of type (4) its fractional Fourier transform takes the form

$$f(\theta,\beta) = \sum_{n=0}^{\infty} a_n\,\mathrm{e}^{\mathrm{i}n\theta}\,\phi_n(\beta)\,. \tag{7}$$

When $\beta = \mathrm{const}$, and denoting $a_n\phi_n(\mathrm{const}) = b_n$, we have

$$f(\theta,\mathrm{const}) = \sum_{n=0}^{\infty} b_n\,\mathrm{e}^{\mathrm{i}n\theta}\,, \tag{8}$$

which means that $f(\theta, \mathrm{const})$ is a periodic function. So, it is possible to interpret the wave propagation as a continuously unfolding fractional Fourier transformation [5].

The eigenfunctions of the ordinary Fourier transform constitute an orthonormal basis for the space of finite-energy signals. The nth Hermite–Gaussian function has the eigenvalue $\lambda_n = \exp(\mathrm{i}n\pi/2)$ and n zero crossings. Since we know that $\exp(\mathrm{i}n\pi/2) = \mathrm{i}^n$, we can also write the Fourier transform as follows:

$$f(\pi/2, \beta) = \sum_{n=0}^{\infty} a_n\, \mathrm{i}^n\, \phi_n(\beta) =$$

$$= \sum_{n=0}^{\infty} a_{4n}\, \phi_{4n}(\beta) - \sum_{n=0}^{\infty} a_{4n+2}\, \phi_{4n+2}(\beta) + \tag{9}$$

$$+\,\mathrm{i}\left[\sum_{n=0}^{\infty} a_{4n+1}\, \phi_{4n+1}(\beta) - \sum_{n=0}^{\infty} a_{4n+3}\, \phi_{4n+3}(\beta)\right].$$

In the similar form we decompose the function $f(\beta) \equiv f(0, \beta)$:

$$f(0, \beta) = \sum_{n=0}^{\infty} a_{4n}\, \phi_{4n}(\beta) + \sum_{n=0}^{\infty} a_{4n+2}\, \phi_{4n+2}(\beta) +$$

$$+ \sum_{n=0}^{\infty} a_{4n+1}\, \phi_{4n+1}(\beta) + \sum_{n=0}^{\infty} a_{4n+3}\, \phi_{4n+3}(\beta)\,. \tag{10}$$

It is also not difficult to see that in the Hermite–Gaussian representation the function and its Fourier transform are different linear combinations of the same functions.

3. Numerical examples

In this section, an eigendecomposition [7] of known functions $\cos(\beta)$ and $\sin(\beta)$ is presented. For convenience of further considerations we define four sets of coefficients:

$$h_0(n) = \sum_{k=0}^{2n} (-1)^k \frac{2^{4n-2k}}{(4n-2k)! \cdot k!}\,, \tag{11}$$

$$h_1(n) = \sum_{k=0}^{2n} (-1)^k \frac{2^{4n-2k+1}}{(4n-2k+1)! \cdot k!}\,, \tag{12}$$

$$h_2(n) = \sum_{k=0}^{2n} (-1)^k \frac{2^{4n-2k+2}}{(4n-2k+2)! \cdot k!}\,, \tag{13}$$

$$h_3(n) = \sum_{k=0}^{2n} (-1)^k \frac{2^{4n-2k+3}}{(4n-2k+3)! \cdot k!}\,. \tag{14}$$

The time-limited $\cos(0, \beta, m)$ and $\sin(0, \beta, m)$ functions are represented using an appropriate couple of the functions

$$f_0(\beta, m) = \sum_{n=0}^{m} \sqrt{\frac{(4n)!}{2^{4n}}}\, h_0(n)\, \phi_{4n}(\beta)\,, \tag{15}$$

$$f_1(\beta, m) = \sum_{n=0}^{m} \sqrt{\frac{(4n+1)!}{2^{4n+1}}}\, h_1(n)\, \phi_{4n+1}(\beta)\,, \tag{16}$$

$$f_2(\beta, m) = \sum_{n=0}^{m} \sqrt{\frac{(4n+2)!}{2^{4n+2}}}\, h_2(n)\, \phi_{4n+2}(\beta)\,, \tag{17}$$

$$f_3(\beta, m) = \sum_{n=0}^{m} \sqrt{\frac{(4n+3)!}{2^{4n+3}}}\, h_3(n)\, \phi_{4n+3}(\beta)\,. \tag{18}$$

For a finite number m in summation (15)–(18), we have expressions for cos and sin functions:

$$\cos(0, \beta, m) = \sqrt{2\frac{\sqrt{\pi}}{\mathrm{e}}}\, [f_0(\beta, m) - f_2(\beta, m)]\,, \tag{19}$$

$$\sin(0, \beta, m) = \sqrt{2\frac{\sqrt{\pi}}{\mathrm{e}}}\, [f_1(\beta, m) - f_3(\beta, m)]\,. \tag{20}$$

Their Fourier transforms are

$$\cos(\pi/2, \beta, m) = \sqrt{2\frac{\sqrt{\pi}}{\mathrm{e}}}\, [f_0(\beta, m) + f_2(\beta, m)]\,, \tag{21}$$

$$\sin(\pi/2, \beta, m) = \sqrt{2\frac{\sqrt{\pi}}{\mathrm{e}}}\, [f_1(\beta, m) + f_3(\beta, m)]\,. \tag{22}$$

At $m = \infty$ $\cos(0, \beta, m) = \cos(\beta)$, $\sin(0, \beta, m) = \sin(\beta)$.

A numerical example of finite-energy $\cos(0, \beta, m)$ and its Fourier transform $F(\cos(0, \beta, m))$ calculated as linear combinations of functions $f_0(\beta, m)$ and $f_2(\beta, m)$ is presented in Fig. 1.

An example of dipole-radiated electric field produced by the Hertzian vector represented by a difference of functions $f_0(\beta, m) - f_2(\beta, m)$ with parameters mapping a finite cos-like pulse (see Fig. 1(a)) in the direction perpendicular to the dipole axis is presented in Fig. 2.

From the result, it is clear that the radiated signal has a component of intermodulation interference. The intermodulation distortion amplitude depends on the number m in summation (15)–(18). It vanishes when

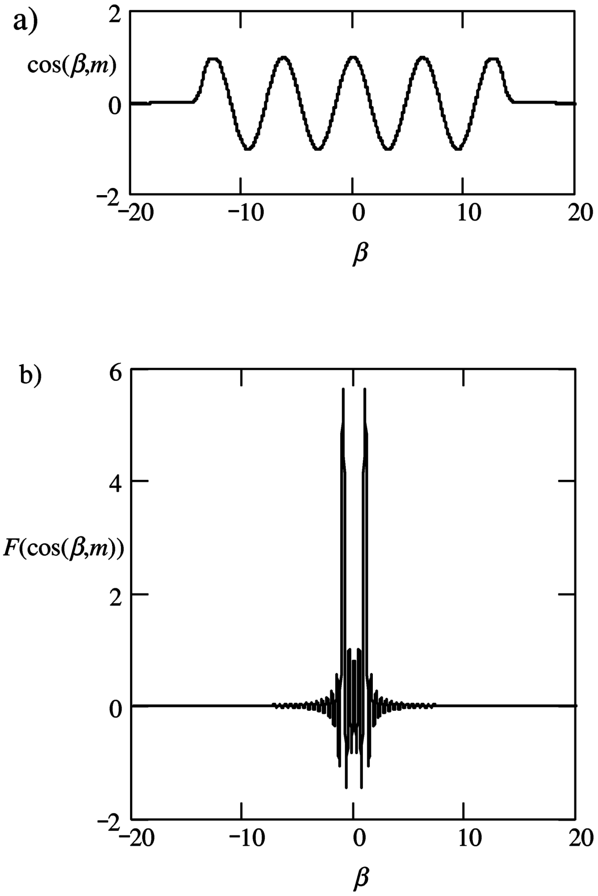

Fig. 1. Representation of (a) $\cos(0, \beta, m)$ and (b) its Fourier transform, $m = 24$.

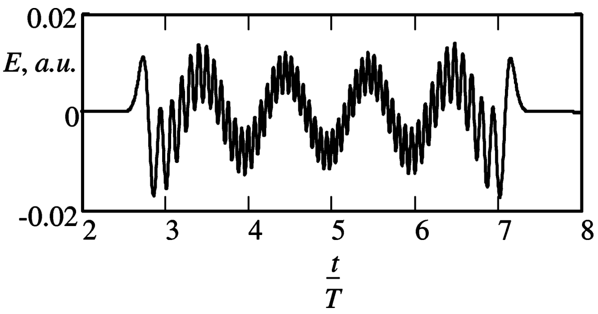

Fig. 2. Dipole-radiated electric field of short finite cos-like excitation at distance $kr = 31$, $m = 24$.

m tends to 0. The interference originates from the imperfection of restricted $\cos(0, \beta, m)$ function. It is also determined by the second order derivative which defines the field in far-zone. For a perfect harmonic signal second order derivative is the same function and no distortion is predicted. The filtering technique is required to eliminate the interference. This topic will be investigated in the future.

4. Conclusion

It has been shown that the propagation process of time and space-limited electromagnetic signals can be described by a couple of proposed functions that are expanded in terms of basis set of the Hermite–Gaussian functions.

The approach also suggests a possible calculation procedure for modelling radiation properties of a device antenna in near and far-zone as a superposition of fields of a finite number of localized dipole sources having various combinations and numbers in the set of discrete functions.

An intermodulation interference in a signal propagation process is predicted which originates from the imperfect representation of a restricted signal and is due to the properties of second order derivatives of functions which define the field.

Acknowledgement

This work was supported in part by the Lithuanian State Science and Studies Foundation under grant No. V-05019.

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DIPOLIO LAUKO IŠRAIŠKA ERMITO IR GAUSO BAZINĖMIS FUNKCIJOMIS

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Santrauka

Tirtas trumpu dipolinio momento impulsu sužadinto taškinio dipolio išspinduliuotas elektromagnetinis laukas. Elektrinis ir magnetinis laukai apskaičiuoti pritaikius žinomą procedūrą iš elektrinio arba magnetinio Herco vektoriaus, kuris šalia dipolio užrašomas Ermito ir Gauso ortonormuotų funkcijų $\phi_n(\beta)$ bazėje. Pritaikius Furjė operatoriaus skleidimo tikrinėmis funkcijomis operaciją, gaunami keturi skleidimo koeficientų rinkiniai. Paėmus atitinkamą rinkinių porą, aprašomas baigtinis erdvėje ar laike dipolio sužadinimas arba jo Furjė atvaizdas. Skaitmeniškai tirti taip sužadinto taškinio dipolio išspinduliuoti laukai. Nustatyta, kad baigtinės trukmės impulsu sužadinto dipolio išspinduliuotas laukas tolimojoje srityje, skirtingai negu jį žadinant harmoniniu lauku, turi intermoduliacinį „triukšmų" sandą. Jo amplitudė priklauso nuo skleidimo narių skaičiaus ir mažėja jam artėjant į nulį. Intermoduliaciniai signalo iškraipymai tolimojoje zonoje susiję su žadinančiojo signalo baigtinumu ir funkcijos, aprašančios laukus, antrųjų išvestinių savybėmis.