

# Atomic Zitterbewegung

M. MERKL<sup>1(a)</sup>, F. E. ZIMMER<sup>1</sup>, G. JUZELIŪNAS<sup>2</sup> and P. ÖHBERG<sup>1</sup>

<sup>1</sup> SUPA, School of Engineering and Physical Sciences, Heriot-Watt University - Edinburgh EH14 4AS, UK, EU

<sup>2</sup> Institute of Theoretical Physics and Astronomy of Vilnius University - A. Gostauto 12, 01108 Vilnius, Lithuania, EU

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**Abstract** – Ultracold atoms which are subject to ultra-relativistic dynamics are investigated. By using optically induced gauge potentials we show that the dynamics of the atoms is governed by a Dirac-type equation. To illustrate this we study the trembling motion of the centre of mass for an effective two-level system, historically called Zitterbewegung. Its origin is described in detail, where in particular the role of the finite width of the atomic wave packets is seen to induce a damping of both the centre of mass dynamics and the dynamics of the populations of the two levels.

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**Introduction.** – Ultracold atoms obeying ultra-relativistic dynamics would be a match made in heaven. An atomic cloud cooled down to nano-kelvin temperatures offers an unprecedented opportunity to study and manipulate a true quantum gas. Relativistic dynamics, on the other hand, seems at first sight incompatible with the concept of an ultracold quantum gas. This is not necessarily the case. It has been noted in earlier works on atomic gases [1–3], trapped ions [4,5], and also in layers of graphene [6,7], that it is indeed possible to study relativistic dynamics for systems which are inherently non-relativistic. Counter-intuitively this is the case in the low-momentum limit.

Already in the early days of quantum mechanics the dynamics of relativistic particles attracted a lot of attention. It was soon realised that a number of counter-intuitive results would follow in the relativistic limit, such as the Zitterbewegung or the Klein paradox. Zitterbewegung has attracted a lot of interest over the years and is continuing to be an active field of research which involves a broad range of physical systems [7–10].

The notion of Zitterbewegung itself has its roots in the work of Schrödinger on the motion of the free particle based on Dirac's relativistic generalization of a wave equation for spin-1/2 particles [11,12]. From these early days on the existence of the Zitterbewegung for relativistic particles has also been the subject of some controversy [13,14].

Zitterbewegung results from the interference of the positive- and negative-energy solutions of the free Dirac equation. The frequency of this interference process is determined by the energy gap between the two possible solution manifolds. In case of the historically first discussed motion of a free electron wave packet this energy gap is on the order of twice the rest energy  $2m_e c^2$  of the electron, *i.e.*, the energy necessary to create an electron-hole pair. For a consistent description in the case of free electrons the concepts of quantum field theory would be necessary. If the Zitterbewegung is driven by a process which corresponds to an energy of the order of the rest mass, then this unfortunately also makes it rather unlikely to observe the trembling motion with real electrons [14].

A number of physical systems, can, however, be described by an effective Dirac equation. For these systems the creation energy of all participating real particles is much larger than the gap energy and thus the processes of particle creation and annihilation can be disregarded. Moreover, the considered *particles* are of a *quasi-particle* nature involving the atomic internal states with different spin projections. The combination of these features allow us to study the phenomenon of Zitterbewegung in regimes far away from its initial discovery —the motion of a free electron.

In this paper we will study the trembling centre of mass motion of an atom characterised by two internal states, which are subject to an off-diagonal matrix gauge potential. In the limit of low momenta and strong gauge

<sup>(a)</sup>E-mail: magm1@hw.ac.uk

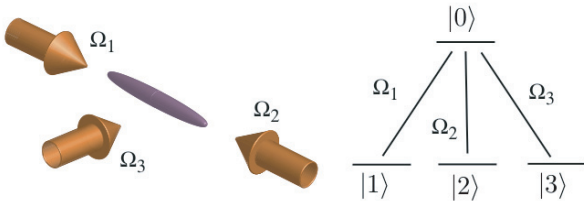


Fig. 1: One possible laser configuration for the tripod system which results in a non-trivial gauge potential for the two corresponding dark states.

fields the dynamics is well described by a Dirac-type equation. In recent papers [2,15–18] it has been shown how atoms with an internal tripod level structure, see fig. 1, may evolve under the influence of an effective non-Abelian vector potential. Here we restrict the motion to only one direction (see also ref. [3] for a two-dimensional description of Zitterbewegung).

The paper is organized as follows. In the following section we will briefly outline the derivation of the gauge potentials for the spin system. As an example of the resulting dynamics in the presence of non-trivial gauge potentials we study the Zitterbewegung for neutral atoms. We discuss the phenomenon of Rabi-type oscillations of the atomic internal states occurring in this context and finally the damping mechanisms due to the finite widths of the wave packets.

**The equation of motion.** – In the following we will assume the motion of the atoms to be restricted to one dimension. We choose our coordinate system such that the  $x$ -axis is aligned along that particular dimension. A gas of ultracold atoms can be considered dynamically one dimensional if the corresponding transversal energy scale given by the transversal trapping frequency is much higher than all other energy scales, such as the temperature or chemical potential in the presence of collisional interactions. In this paper we neglect collisions between the atoms which is justified if we consider sufficiently dilute gases.

To the effectively one-dimensional cloud of cold atoms we apply the scheme for inducing non-Abelian gauge potentials as presented in [16], and obtain in the limit of low momenta a *quasi-relativistic* situation as shown in [2]. For this purpose, we consider the adiabatic motion of atoms in the presence of three resonant laser beams. The technique is remarkably versatile and offers the possibility to shape the gauge potentials quite freely. Various possibilities exist for creating non-trivial equations of motions [16]. Here we have chosen a laser configuration where two of the beams have the same intensity but counter-propagate. The third laser beam has a different intensity compared to the two other laser beams. Its wave vector is chosen to be perpendicular to the axis defined by the propagation direction of laser 1 or 2. The configuration is depicted in fig. 1.

By defining the total Rabi frequency  $\Omega = \sqrt{\sum_{n=1}^3 |\Omega_n|^2}$  and the mixing angle  $\theta$  from  $\tan \theta = \sqrt{|\Omega_1|^2 + |\Omega_2|^2} / |\Omega_3|$ , we can write the Rabi frequencies of the participating laser fields in the following form:  $\Omega_1 = \Omega \sin \theta e^{-i\kappa x} / \sqrt{2}$ ,  $\Omega_2 = \Omega \sin \theta e^{i\kappa x} / \sqrt{2}$  and  $\Omega_3 = \Omega \cos \theta e^{-i\kappa y}$ . Applying this notation we find that in the interaction picture the Hamiltonian is given by

$$\hat{H}_{int} = -\hbar (\Omega_1 |0\rangle\langle 1| + \Omega_2 |0\rangle\langle 2| + \Omega_3 |0\rangle\langle 3|) + h.a. \quad (1)$$

The Hamiltonian  $\hat{H}_{int}$  yields two dark states  $|D_i\rangle$ ,  $i = 1, 2$ , which contain no contribution from the excited state  $|0\rangle$ :

$$|D_1\rangle = \frac{1}{\sqrt{2}} e^{-i\kappa y} (e^{i\kappa x} |1\rangle - e^{-i\kappa x} |2\rangle), \quad (2)$$

$$|D_2\rangle = \frac{1}{\sqrt{2}} e^{-i\kappa y} (\cos \theta (e^{i\kappa x} |1\rangle + e^{-i\kappa x} |2\rangle) - \sin \theta |3\rangle). \quad (3)$$

Both dark states are eigenstates of  $\hat{H}_{int}$  with zero eigenenergy. They depend on the position due to the spatial dependence of the Rabi frequencies  $\Omega_i$ .

The bright state  $|B\rangle \sim \Omega_1^* |1\rangle + \Omega_2^* |2\rangle + \Omega_3^* |3\rangle$  is coupled to the excited state  $|0\rangle$  with the Rabi frequency  $\Omega$  and therefore separated from the dark states by energies  $\pm \hbar \Omega$ . If  $|\Omega|$  is large compared to any two-photon detuning or Doppler shifts due to the atomic motion, we can neglect transitions out of the dark states, *i.e.*, we use the adiabatic approximation. In this limit it is sufficient to expand the general state vector  $|\chi\rangle$  of the quantum system in the dark state basis

$$|\chi(\mathbf{r}, t)\rangle = \sum_{i=1}^2 \Psi_i(\mathbf{r}, t) |D_i(\mathbf{r})\rangle, \quad (4)$$

where the expansion coefficients  $\Psi_i(\mathbf{r}, t)$  are the wave functions for the centre of mass motion of the atoms in the dark state  $i$ . By collecting the wave functions in the spinor

$$\bar{\Psi} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \quad (5)$$

we find that the latter obeys the effective Schrödinger equation [16]

$$i\hbar \frac{\partial}{\partial t} \bar{\Psi} = \left[ \frac{1}{2m} (\mathbf{p}_x - \hat{\mathbf{A}})^2 + \hat{V} + \hat{\Phi} \right] \bar{\Psi}, \quad (6)$$

where  $\mathbf{p}_x$  denotes the momentum along the  $x$ -axis and  $m$  is the atomic mass. Here  $\hat{\mathbf{A}}$  is an effective vector potential matrix, also called the Mead-Berry connection [19,20] and  $\hat{V}$  and  $\hat{\Phi}$  are effective scalar potentials matrices. The gauge potentials  $\mathbf{A}_{n,m} = i\hbar \langle D_n(\mathbf{r}) | \nabla D_m(\mathbf{r}) \rangle$  and  $\Phi_{n,m} = \frac{\hbar^2}{2m} \langle D_n(\mathbf{r}) | \nabla B(\mathbf{r}) \rangle \langle B(\mathbf{r}) | \nabla D_m(\mathbf{r}) \rangle$  emerge due to the spatial dependence of the dark states. The additional scalar potential is defined by  $V_{n,m} = \langle D_n(\mathbf{r}) | \hat{V} | D_m(\mathbf{r}) \rangle$  with  $\hat{V} = \sum_{j=1}^3 V_j(\mathbf{r}) |j\rangle\langle j|$  and  $V_j(\mathbf{r})$  being the trapping potential for atoms in the bare state  $j$ .

With the setup presented in fig. 1 these potentials take in the  $x$ -direction the form

$$\hat{\mathbf{A}} = -\hbar\kappa \begin{pmatrix} 0 & \mathbf{e}_x \cos \theta \\ \mathbf{e}_x \cos \theta & 0 \end{pmatrix} = -\hbar\kappa' \sigma_x \mathbf{e}_x, \quad (7)$$

$$\hat{\Phi} = \frac{\hbar^2 \kappa^2}{2m} \begin{pmatrix} \sin^2 \theta & 0 \\ 0 & \sin^2(2\theta)/4 \end{pmatrix}, \quad (8)$$

$$\hat{V} = \begin{pmatrix} V_1 & 0 \\ 0 & V_1 \cos^2 \theta + V_3 \sin^2 \theta \end{pmatrix}, \quad (9)$$

where we have introduced the notation  $\kappa' = \kappa \cos(\theta)$  and assumed that the external trapping potentials for the first two atomic states are the same, *i.e.*  $V_1 = V_2$ . In addition, the external trap in the transversal direction is assumed to dominate over any effective gauge potential in this direction. Hence, we can use an effectively one-dimensional equation of motion.

**The Dirac limit.** – In the considered setup the presence of an energy gap is necessary in order to observe Zitterbewegung. Such a gap is obtained by different, but constant, trapping potentials  $V_1$  and  $V_3$  which can be altered by detuning the corresponding lasers from the atomic transitions. Alternatively, the intensity ratio of the laser beams can be used to adjust the scalar potential.

It is convenient to introduce the following notation:

$$V_z = \frac{1}{2} [V_{11} + \Phi_{11} - (V_{22} + \Phi_{22})] \quad (10)$$

and shift the zero level of energy. The trapping potential then reads

$$\hat{V} + \hat{\Phi} = V_z \sigma_z = V_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (11)$$

where  $\sigma_z$  is one of the Pauli spin matrices. In the limit of low momenta, *i.e.*,  $|p| \ll \hbar\kappa$ , we can neglect in eq. (6) the kinetic energy term and are left with an effective Dirac equation

$$i\hbar\partial_t \bar{\Psi} = \left[ -\frac{\mathbf{A} \cdot \mathbf{p}_x}{m} + \frac{\mathbf{A}^2}{2m} + V_z \sigma_z \right] \bar{\Psi} \quad (12)$$

$$= H_{DA} \bar{\Psi} = [\tilde{c} \sigma_x \mathbf{p}_x + V_z \sigma_z] \bar{\Psi}. \quad (13)$$

Here  $\tilde{c} = \frac{\hbar\kappa}{m}$  is a recoil velocity, which is typically on the order of cm/s for alkali atoms and photons in the optical spectral region. The transition from the Schrödinger to the Dirac limit, in eq. (13), is most clearly justified by considering a wave packet with increasing width as shown in fig. 2 [2]. Equation (13) is the starting point of our main discussion. We note that the  $\mathbf{A}^2$  can be absorbed into the potential term and we are hence left with an equation which resembles the Dirac equation for a free relativistic particle with the rest energy substituted by the potential energy difference between the two levels.

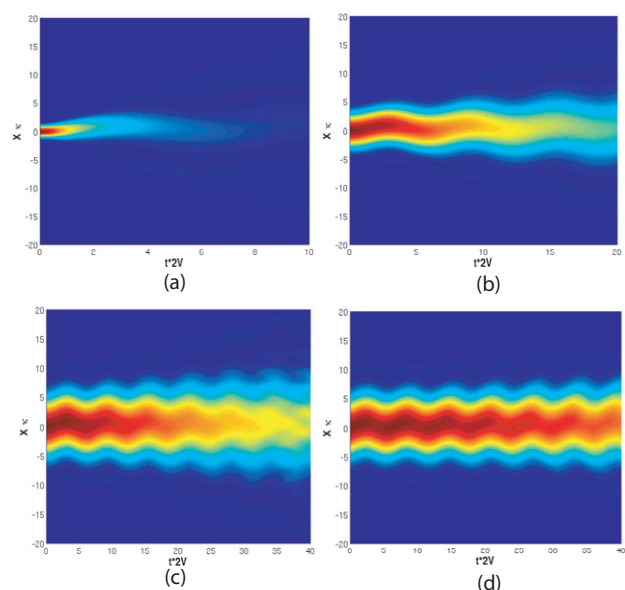


Fig. 2: Density plot showing the Dirac limit and the role of the initial width of the wave packets. Figures (a)–(c) display the full Schrödinger dynamics using eq. (6) for initial Gaussian states with increasing width  $\sigma$ . This results in a sharper momentum distribution with  $|p| \ll \hbar\kappa$  increasingly fulfilled. The pure Dirac case is shown in (d) for comparison. The dynamics in (a)–(d) shows in addition to the Zitterbewegung also a damping.

**Zitterbewegung.** – In most textbooks [21,22] Zitterbewegung is derived by solving the Heisenberg equation for the position operator. For a free particle with rest mass  $m$ , the Dirac Hamiltonian  $H_D$  containing the speed of light  $c$ ,

$$H_D = c\alpha\mathbf{p} + \beta mc^2, \quad (14)$$

is used to obtain the time dependence for the position operator  $\hat{\mathbf{x}}$ . By using the anticommutation properties of the Dirac  $\alpha$  and  $\beta$  matrices one finds

$$\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}(0) + H_D^{-1} c^2 \mathbf{p} t - \frac{i\hbar c}{2} H_D^{-1} \left( e^{-2iH_D t/\hbar} - 1 \right) \left( \alpha(0) - c\mathbf{p} H_D^{-1} \right). \quad (15)$$

In eq. (15) the first and second term describe a motion which is linear in time, while the third term gives an oscillating contribution, the Zitterbewegung. To observe this trembling motion an initial 4-component spinor state needs to contain positive and negative energy solution parts as the  $\alpha$  matrix is mixing these. The frequency of the oscillating term can be estimated in the particles rest frame as  $2mc^2/\hbar$ . This typically large energy is the energy difference between a particle and antiparticle. To obtain Zitterbewegung in cold quantum gases we emphasise that the two dark states described by eq. (13) are not a particle-antiparticle pair, but they are still separated by an energy gap which is generated by the constant potential term in eq. (11). Different to previous work [2] the Dirac Hamiltonian  $H_{DA}$  now contains a term which corresponds to an effective rest mass.

**Exact solutions in the Schrödinger limit.** – Before discussing the wave packet dynamics we briefly summarise the solution of the Schrödinger equation (6) with the gauge potentials from eqs. (7)–(9). The general solution can readily be written down in momentum space,

$$\bar{\Psi}(k, \tau) = e^{-i(k^2 + 2\sigma_x k + \tilde{V}_z \sigma_z)\tau} \bar{\Psi}(k, 0), \quad (16)$$

where now the dimensionless  $k$  is expressed in units of  $\kappa'$  and the time  $\tau$  in units of  $2m/\hbar\kappa'^2$ . If we choose a Gaussian momentum distribution with a width  $\Delta$  for the initial state,

$$\bar{\Psi}(k, 0) = \frac{1}{\sqrt{\Delta\sqrt{\pi}}} e^{-(k-k_0)^2/2\Delta^2} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad (17)$$

we obtain an exact time-dependent solution of the form

$$\bar{\Psi}(k, \tau) = \frac{1}{\sqrt{\Delta\sqrt{\pi}}} e^{-\frac{(k-k_0)^2}{2\Delta^2} + i(k^2+1)\tau} \times \begin{pmatrix} c_1 \cos(\omega_k \tau) + \frac{i(c_1 \tilde{V}_z + c_2 2k)}{\omega_k} \sin(\omega_k \tau) \\ c_2 \cos(\omega_k \tau) - \frac{i(c_2 \tilde{V}_z - c_1 2k)}{\omega_k} \sin(\omega_k \tau) \end{pmatrix}, \quad (18)$$

where we have introduced the  $k$ -dependent frequency

$$\omega_k = \sqrt{4k^2 + \tilde{V}_z^2}. \quad (19)$$

With the solution  $\bar{\Psi}(k, \tau)$  we can calculate the centre of mass motion for the two-component wave packet where we use the standard definition of the density,  $\rho(k, \tau) = |\Psi_1(k, \tau)|^2 + |\Psi_2(k, \tau)|^2$ , or the dynamics of any other quantity depending upon the two dark states.

*Dark state dynamics.* Our system with two degenerate dark states shows not only a relativistic behaviour, but also properties familiar from two-level systems in quantum optics [23]. In order to see this in more detail we write the spinor  $\bar{\Psi}(x, t)$  as a combination of slowly varying envelopes,  $\phi_i(x, t)$   $i = 1, 2$ , and coefficients which describe the population of the two dark states,

$$\bar{\Psi} = \begin{pmatrix} \phi_1(x)c_1(t) \\ \phi_2(x)c_2(t) \end{pmatrix}. \quad (20)$$

The spatial shape  $\phi_i(x, t)$  should change much slower than the population  $c_i(t)$  of the  $i$ -th component of the spinor such that we can neglect all its derivatives with respect to time. The solutions are normalised according to  $\langle \phi_i | \phi_i \rangle = 1$  and  $|c_1|^2 + |c_2|^2 = 1$ . After inserting the ansatz from eq. (20) into eq. (13) we obtain a set of coupled differential equations for the coefficients:

$$i \begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = \begin{pmatrix} V_{z1} & \tilde{\Omega} \\ \tilde{\Omega}^* & V_{z2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad (21)$$

where

$$\tilde{\Omega} = \frac{\tilde{c}}{\hbar} \langle \phi_2 | \mathbf{P}_x | \phi_1 \rangle, \quad (22)$$

and

$$V_{zi} = \langle \phi_i | V_z | \phi_i \rangle / \hbar. \quad (23)$$

The two spin components are coupled, hence the solutions to eq. (21) will show population oscillations between the two dark states with a frequency

$$\omega_R^2 = |\tilde{\Omega}|^2 + \frac{1}{4} (V_{z1} - V_{z2})^2. \quad (24)$$

For a vanishing overlap integral  $\tilde{\Omega}$ , the coupling between the spin components in eq. (21) becomes zero and we expect no oscillations of the populations. A non-vanishing effective Rabi frequency  $\tilde{\Omega}$  can be achieved if the initial state has a non-zero momentum of the form

$$\phi_i(x) = e^{-x^2/\sigma^2 + ik_0 x} \quad (25)$$

with a width  $\sigma$  and momentum  $k_0$ , which will consequently result in population transfer between the two dark states. A simulation of this situation is shown in fig. 3(f) and fig. 4. The splitting of the wave packets and their motion in opposite directions can be well understood by considering the dispersion relation of eqs. (6) or (13). For  $k_0 \neq 0$  the wave packets experience different group velocities from the different dispersion branches.

To illustrate in more detail the phenomenon of Rabi oscillations we examine again the exact solution in eq. (18) and choose an initial state with  $c_1 = c_2 = 1/\sqrt{2}$ . The population difference,  $\Delta N(t) = |\psi_1(t)|^2 - |\psi_2(t)|^2$ , can easily be calculated in the limit  $\Delta = 0$ . In this limit the Gaussian initial state turns into a representation of the delta function,

$$\Delta N(\tau) = \frac{4k_0 \tilde{V}_z}{\sqrt{4k_0^2 + \tilde{V}_z^2}} \sin^2 \left( \tau \sqrt{4k_0^2 + \tilde{V}_z^2} \right). \quad (26)$$

From this result we see the importance of the initial momentum  $k_0$ . For  $k_0 = 0$  there is no transfer of population between the dark states, whereas for a non-zero initial momentum the amplitude of the population oscillation is proportional to  $k_0$ . In addition, the frequency  $\sqrt{4k_0^2 + \tilde{V}_z^2}$  is  $k_0$ -dependent as well.

This behavior is shown for a finite width of the wave packets in fig. 4 where the width was chosen such that the dynamics takes place in the Dirac limit. In this case, an attenuation of the amplitude of the dark state population difference also occurs. The Rabi oscillations are accompanied by the disappearance of the Zitterbewegung; an effect we will study in the next section in more detail.

*The centre of mass dynamics.* The transient nature of Zitterbewegung has already been studied in effective relativistic systems such as mono- and bilayer graphite [6] and ultracold atoms [3]. These studies differ, however, from the present one in several respects. Firstly, refs. [3,6] consider two-dimensional systems and secondly, they found that a non-zero momentum in one direction leads to Zitterbewegung in the perpendicular direction. This is in contrast to our studies where the Zitterbewegung

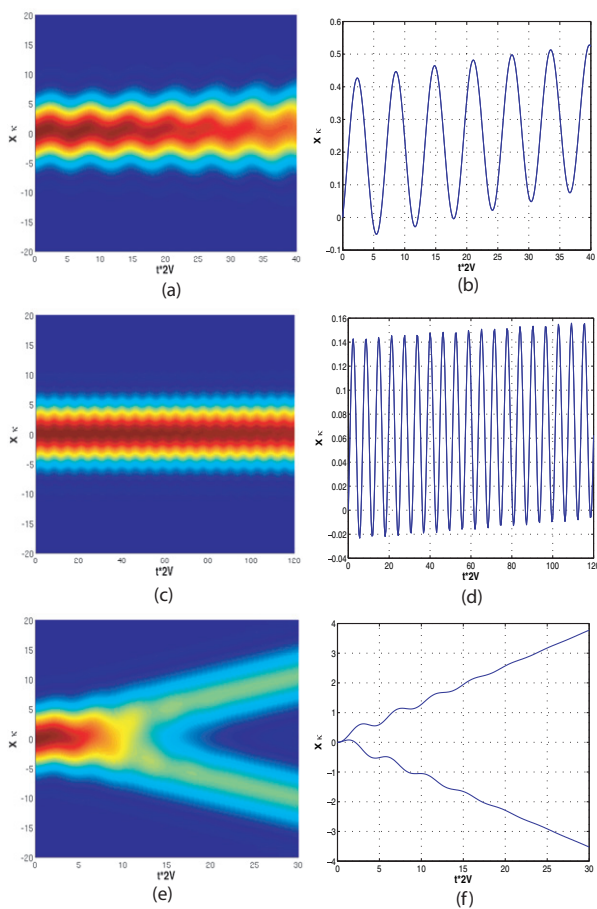


Fig. 3: Left column: the density as a function of time shows Zitterbewegung for different energy gaps ((a) with  $V_z = \frac{\hbar^2 \kappa^2}{2m}$  and (c) with  $V_z = 3\frac{\hbar^2 \kappa^2}{2m}$ ) and in (e) with an initial momentum  $k_0 = \kappa'$ . Right column: the centre of mass shows the expected oscillation ((b) and (d)) with an upward drift. With an initial momentum kick the behaviour is different compared to (a) and (c), as can be seen in (e) and (f), where the Zitterbewegung breaks down after a few oscillations as the two states are moving in different directions. The initial spinor was in (a)–(d)  $(1, 1)^T/\sqrt{2}$  and in (e) and (f)  $(1, e^{i\pi/4})^T/\sqrt{2}$ .

is induced by the potential term in eq. (13) (compare with eq. (23) in [6] or eq. (11) in [3]). Apart from the non-vanishing initial momentum leading inevitably to a vanishing of the interference effect, a finite width of the wave packets also leads to an attenuation of the Zitterbewegung. As shown in fig. 2 the attenuation occurs in both limits, *i.e.*, the Schrödinger and the Dirac limit.

In the following we analyse the exact solution (18) and study the role of the finite width of the wave packets on the Zitterbewegung. To this end we consider the center of mass of the wave packets, *i.e.*

$$\langle x(\tau) \rangle = i \int_{-\infty}^{\infty} dk \bar{\Psi}^\dagger(k, \tau) \partial_k \bar{\Psi}(k, \tau) = \frac{1}{\Delta \sqrt{\pi}} \int_{-\infty}^{\infty} dk e^{-\frac{k^2}{\Delta^2}} \left( \frac{4k^2}{\omega_k^2} \tau + \frac{\tilde{V}_z^2}{\omega_k^3} \sin(2\omega_k \tau) \right), \quad (27)$$

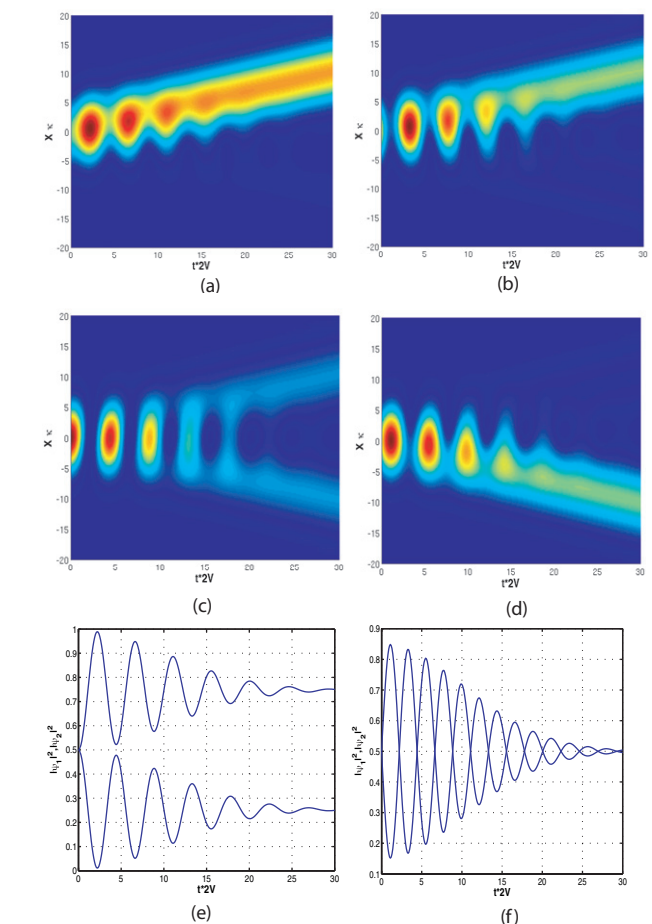


Fig. 4: Rabi type oscillations superimposing the Zitterbewegung can be seen for an initial preparation with non-zero momentum  $k_0 = \kappa'$ . The dynamics is sensitive to the initial conditions. Panels (a) and (b) show  $|\Psi_1(t, x)|^2$  for the initial spinor  $(1, 1)^T/\sqrt{2}$  and  $(1, e^{i\pi/4})^T/\sqrt{2}$ , respectively, whereas panels (c) and (d) show  $|\Psi_2(t, x)|^2$  for the initial spinor  $(1, 1)^T/\sqrt{2}$  and  $(1, e^{i\pi/4})^T/\sqrt{2}$ . The population of the dark states is depicted in (e) with  $(1, 1)^T/\sqrt{2}$  and in (f) with  $(1, e^{i\pi/4})^T/\sqrt{2}$  as initial states.

where we have assumed  $k_0 = 0$ . From the first term under the integral sign we obtain a drift term for the centre of mass,

$$x_d = \tau \left[ 1 - \sqrt{\pi} \frac{\tilde{V}_z}{\Delta} e^{\frac{\tilde{V}_z^2}{\Delta^2}} \text{Erfc} \left( \frac{\tilde{V}_z}{\Delta} \right) \right], \quad (28)$$

where Erfc is the complementary Error function. In the limit of  $\tilde{V}_z/\Delta \gg 1$ , and using the asymptotic expansion of the Error function,

$$\text{Erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \left( 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{(2x^2)^n} \right), \quad (29)$$

we obtain a reduced drift as a function of *increasing*  $\tilde{V}_z/\Delta$ . This is a finite-size effect and stems from the finite width of the wave packet. The drift motion has already been identified in our numerical analysis as shown in fig. 3.

The second term under the integral in (27) stands for the Zitterbewegung. This is an effect due to the  $\hat{\mathbf{A}}\mathbf{p}_x$ -term in the Schrödinger equation (6) in contrast to the mechanism in [24]. Again, in the limit  $\tilde{V}_z/\Delta \gg 1$ , the integral can be readily calculated and gives for the oscillating part

$$x_z = \frac{1}{\tilde{V}_z} \frac{\sin\left(2\tilde{V}_z\tau + \frac{1}{2}\arctan\left(\frac{\Delta^2}{4\tilde{V}_z}\tau\right)\right)}{\left(1 + \frac{\Delta^4}{16\tilde{V}_z^2}\tau^2\right)^{1/4}}. \quad (30)$$

From this expression we see that a spread in the momentum distribution will cause a damping also for the oscillating term of the centre of mass. The damping of the Zitterbewegung is relatively slow, but inevitable. This can be explained by envisaging a collection of oscillators each with a different frequency which is  $k$ -dependent. In this case the centre of mass will show a damping if the revival time is infinite. The underlying equation is after all the Schrödinger equation, and only in the limit  $\Delta = 0$  should we strictly speaking use a Dirac-type equation. With the full Schrödinger equation a free wave packet will always expand, albeit slowly if  $\Delta$  is small, and hence will also show a damped Zitterbewegung.

For a typical alkali atom such as  $^{87}\text{Rb}$  with a wave packet width of  $10\ \mu\text{m}$  one would get  $\Delta^2\tau/4\tilde{V}_z > 1$  for times larger than 1 ms, with a centre of mass oscillation frequency of the order of 1 kHz and a corresponding amplitude of the order of two microns. Hence a broad wave packet as initial state would favour the detection of the Zitterbewegung. The experimental setup for observing these effects would be remarkably simple. The dark states need to be prepared, but the rest is free expansion.

**Conclusions.** – In this letter we have showed using ultracold atoms how Zitterbewegung, known from relativistic physics, is a generic phenomenon which will naturally occur in systems with degenerate eigenstates. The additional effects such as drift, attenuation and Rabi-type oscillations have been discussed. The role of initial momenta and the width of the wave packets has been investigated. Interestingly, the atomic scenario offers a number of new possibilities. We are now in a position, for instance, to study a system which would correspond to a confined Dirac particle by introducing external atomic potentials, either by optical or magnetical means, for the atoms. If the external trap is weak compared to the  $\tilde{V}_z$ , its influence on the wave packet dynamics can be deduced from the dispersion relation. In the opposite limit the trap dynamics dominates with reduced Zitterbewegung. This consequently leads us to ponder whether the present system will show Bose-Einstein condensation, and, if so, what will such a quantum state look like [25]. In this context interactions between the atoms will play an important role, where one would be faced with a non-linear Dirac-type equation to describe the dynamics.

The source of the non-linearity is, however, non-trivial due to the underlying collisions between the two dark states.

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