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Herding model and 1/f noise

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Abstract – We provide evidence that for some values of the parameters a simple agent-based model, describing herding behavior, yields signals with $1/f$ power spectral density. We derive a non-linear stochastic differential equation for the ratio of number of agents and show, that it has the form proposed earlier for modeling of $1/f^{\beta}$ noise with different exponents β. The non-linear terms in the transition probabilities, quantifying the herding behavior, are crucial to the appearance of $1/f$ noise. Thus, the herding dynamics can be seen as a microscopic explanation of the proposed non-linear stochastic differential equations generating signals with $1/f^{\beta}$ spectrum. We also consider the possible feedback of macroscopic state on microscopic transition probabilities strengthening the non-linearity of equations and providing more opportunities in the modeling of processes exhibiting power-law statistics.

Introduction. – Kirman's seminal herding model was introduced in refs. [1,2]. This is a simple stochastic model of information transmission initially designed to explain the herding behavior in ant colonies, gathering food from two identical sources located in their neighborhood. Kirman noticed that entomologists and economists observe similar patterns in rather different systems. If there are two identical food sources available to ants in a colony, majority of ants still tend to use only a single food source at any given time. Furthermore, switches to the new food source occur despite the fact that food sources remain identical [3]. Human crowd behavior tends to be quite similar, at least in the statistical sense. There are observations that the majority of people tend to choose more popular products, than less popular ones, despite both being of a similar quality. The article [2] also cites numerous works, which speculate that herding behavior might be related to the fluctuations of asset price. In Kirman's formalization the switching probabilities do not depend on the source, thus the probability distribution of ant's visiting times at both mangers is symmetric.

Kirman's model and similar approaches have been applied as models of herding and contagion phenomena in financial markets [1,4,5]. In ref. [6] the equilibrium distribution of the related discrete-time stochastic process within the more general theoretical framework of Polya urn processes has been derived. The associated Fokker-Planck equation for the pertinent continuous symmetric dynamic process has been derived and solved in ref. [7]. The parameters of Kirman's herding model applied to the description of financial markets were estimated in ref. [8] by the introduction of a simulated moment approach extracting two key parameters of the model via matching of the empirical kurtosis and the first autocorrelation coefficient of squared returns. A direct estimation of the parameters of a related agent-based model, based on a closed-form solution of the unconditional distribution of returns, has been proposed in ref. [9]. Kirman's model was generalized in ref. [10]. It is worth noticing that the appropriate agent-based models can yield emergence the power-law scaling, long-range correlations, (multi)fractality and fat tails (see, *e.g.*, [11,12] and references herein), however the omnipresent $1/f$ noise have not yet been revealed in such approach.

The phrases "$1/f$ noise", "$1/f$ fluctuations", and "flicker noise" refer to the phenomenon, having the power spectral density at low frequencies f of signals of the form $S(f) \sim 1/f^{\beta}$, with β being a system-dependent parameter. Power-law distributions of spectra of signals with $0.5 < \beta < 1.5$, as well as scaling behavior in general, are ubiquitous in physics and in many other fields, including natural phenomena, human activities, traffics in computer

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networks and financial markets [13–23]. This subject has been a hot research topic for many decades (see, *e.g.*, a bibliographic list of papers by Li [24], and a short review in Scholarpedia [13]). Despite the numerous models and theories proposed since its discovery more than 80 years ago [25,26], the subject of $1/f$ noise remains still open for new discoveries. Most of the models and theories are not universal because of the assumptions specific to the problem under consideration. In 1987 Bak *et al.* [27] introduced the notion of self-organized criticality (SOC) with the motivation to explain the universality of $1/f$ noise, as well. Although the paper [27] is the most cited paper in the field of $1/f$ noise problems, it was shown later on [28,29] that the mechanism proposed in [27] results in $1/f^{\beta}$ fluctuations with $1.5<\beta<2$ and does not explain the omnipresence of $1/f$ noise. On the other hand, we can point to a recent paper [30] where an example of $1/f$ noise in the classical sandpile model has been provided. A short categorization of the theories and models of $1/f$ noise is presented in the introduction of the paper [31].

Recently, the non-linear stochastic differential equations (SDEs) generating signals with $1/f$ noise were obtained in refs. [32,33] (see also recent paper [31]), starting from the point process model of $1/f$ noise [34–39]. Nonlinear SDEs provide macroscopic description of a complex system. Microscopic, agent-based reasoning of equations exhibiting $1/f$ noise can yield further insights into the behavior of the system. In this paper we show that it is possible to obtain non-linear SDE of the form of refs. [32,33] starting from the agent-based herding model. Thus, it is possible to show analytically that the non-linear nature of herding interactions and appropriate choice of variable results in $1/f$ fluctuations.

The herding model. – In papers [1,2,4] Kirman employed a simple model to describe the behavior of a multitude of heterogeneous interacting agents. In the model the dynamic evolution is described as a Markov chain. There is a fixed number N of agents, each of them being in state 1 or in state 2. The number of agents in the first state is denoted by n, and the number in the second state by $N-n$. The core of Kirman's model is the pairwise interaction governing the transition of the agents between the two states. Neither the probability of following another companion nor the success in recruiting companions depend on the outcome of previous meetings. The lack of memory of the agents is the crucial assumption to formalize the population dynamics as a Markov process. Describing the dynamics as a jump Markov process in continuous time, the transition probabilities per unit time are given by

$$p(n\rightarrow n+1)\equiv p^{+}(n)=(N-n)(\sigma_1+hn), \tag{1}$$

$$p(n\rightarrow n-1)\equiv p^{-}(n)=n(\sigma_2+h(N-n)). \tag{2}$$

The above probabilities define a one-step stochastic process [40]. The constants σ_1 and σ_2 describe the idiosyncratic propensity to change the state, while the term h describes the herding tendency. We allow the two constant parameters σ_1 and σ_2 to assume different values, generating asymmetric behavior. The non-linearity in the transition probabilities (1) and (2) constitutes a crucial ingredient of the model: the presence of non-linear terms, is the imprint of interactions among agents. The linear terms would imply independence of the behavior of the agents.

The transition rates (1) and (2) describe a scenario where the interaction among agents does not depend on the fraction of agents in the alternative states, but rather on the overall number of such agents. Such a choice makes the transition rates non-extensive, the connectivity between agents increases with the number of agents N. The interactions have a global character, the range of the correlations involves a macroscopic fraction of agents. This means that temporal correlations in the level of fluctuations might be observed for any system size. We will show further that such non-linear terms in the transition probabilities leads to $1/f$ behavior of the power spectral density.

The transition probabilities imply the Master equation for the probability $P_n(t)$ to find n agents in the state 1 at time t [40]:

$$\begin{aligned}\frac{\partial}{\partial t}P_n &= p^{+}(n-1)P_{n-1}+p^{-}(n+1)P_{n+1}\\ &\quad -(p^{+}(n)+p^{-}(n))P_n.\end{aligned} \tag{3}$$

For large enough N we can represent the group dynamics by a continuous variable $x=n/N$. Using standard methods from ref. [40], a Fokker-Planck equation is derived from the Master equation (3) assuming that N is large and neglecting the terms of the order of $1/N^2$:

$$\begin{aligned}\frac{\partial}{\partial t}P_x(x,t) &= -\frac{\partial}{\partial x}h(\varepsilon_1(1-x)-\varepsilon_2 x)P_x(x,t)\\ &+\frac{1}{2}\frac{\partial^2}{\partial x^2}h\left(2x(1-x)+\frac{\varepsilon_1}{N}(1-x)+\frac{\varepsilon_2}{N}x\right)P_x(x,t),\end{aligned} \tag{4}$$

where $\varepsilon_1\equiv\sigma_1/h$, $\varepsilon_2\equiv\sigma_2/h$ are scaled parameters. In the following we will ignore the terms of the order of $1/N$ in the diffusion term in eq. (4), assuming that variable x is not too close to the boundaries $x=0$ and $x=1$, *i.e.*, $x\gg\varepsilon_1/N$ and $1-x\gg\varepsilon_2/N$. In addition we assume that $\varepsilon_1,\varepsilon_2>0$. Thus the Fokker-Planck equation for the herding model has the form

$$\begin{aligned}\frac{\partial}{\partial t}P_x(x,t) &= -\frac{\partial}{\partial x}h(\varepsilon_1(1-x)-\varepsilon_2 x)P_x(x,t)\\ &\quad +\frac{\partial^2}{\partial x^2}hx(1-x)P_x(x,t).\end{aligned} \tag{5}$$

This Fokker-Planck equation corresponds to the stochastic differential equation

$$\mathrm{d}x=h(\varepsilon_1(1-x)-\varepsilon_2 x)\mathrm{d}t+\sqrt{2hx(1-x)}\mathrm{d}W, \tag{6}$$

where W is a standard Wiener process (the Brownian motion). The steady-state solution of eq. (5) has the form

$$P_0(x) = \frac{\Gamma(\varepsilon_1+\varepsilon_2)}{\Gamma(\varepsilon_1)\Gamma(\varepsilon_2)} x^{\varepsilon_1-1}(1-x)^{\varepsilon_2-1}. \tag{7}$$

Equations (5)–(7) were obtained in ref. [9].

Using $\varepsilon_1=\varepsilon_2$ in eq. (7) we recover the equilibrium distribution as in ref. [2]. The distribution (7) exhibits a unique mode if both parameters take a value larger than 1, while it shows bi-modality for the case $\varepsilon_1, \varepsilon_2 < 1$. Furthermore, the distribution shows a monotonic behavior if one parameter is larger than 1 and the other smaller than 1.

Non-linear stochastic differential equation generating signals with $1/f^\beta$ noise. – Starting from the point process model, proposed and analyzed in refs. [34–39], the non-linear SDEs generating processes with $1/f^\beta$ noise are derived [31–33]. The general expression for the proposed class of Itô SDEs is

$$\mathrm{d}x = \sigma^2\left(\eta - \frac{1}{2}\lambda\right) x^{2\eta-1}\mathrm{d}t + \sigma x^\eta \mathrm{d}W. \tag{8}$$

Here x is the signal, $\eta \neq 1$ is the power-law exponent of the multiplicative noise, λ defines the behavior of stationary probability distribution, and W is a standard Wiener process (the Brownian motion). The Fokker-Planck equation corresponding to SDE (8) gives the power-law probability density function (PDF) of the signal intensity $P_0(x) \sim x^{-\lambda}$ with the exponent λ. In refs. [33,39] it was shown that SDE (8) generates signals with power spectral density

$$S(f) \sim \frac{1}{f^\beta}, \qquad \beta = 1 + \frac{\lambda-3}{2(\eta-1)}. \tag{9}$$

The non-linear SDE (8) has the simplest form of the multiplicative noise term, $\sigma x^\eta\,\mathrm{d}W$. Multiplicative equations with the drift coefficient proportional to the Stratonovich drift correction for transformation from the Stratonovich to the Itô stochastic equation [41] generate signals with the power-law distributions [31]. Equation (8) is of such a type and has the stationary PDF of the power-law form. The connection of the power spectral density of the signal generated by SDE (8) with the behavior of the eigenvalues of the corresponding Fokker-Planck equation was analyzed in ref. [42]. In ref. [43] it is shown that $1/f^\beta$ noise is equivalent to a Markovian eigenstructure power relation.

SDE (8) exhibits the following scaling property: changing the stochastic variable from x to a scaled variable $x' = ax$ changes the time scale of the equation to $t' = a^{2(1-\eta)}t$, leaving the from of the equation unchanged. This scaling property is one of the reasons for the appearance of the $1/f^\beta$ power spectral density.

Another remarkable property of SDE (8) is the behavior under transformation of the variable x: if instead of x we introduce

$$y = x^\alpha, \tag{10}$$

then from eq. (8) we get the equation of the same type

$$\mathrm{d}y = \sigma'^2\left(\eta' - \frac{\lambda'}{2}\right) y^{2\eta'-1}\mathrm{d}t + \sigma' y^{\eta'}\mathrm{d}W, \tag{11}$$

only with different parameters $\sigma' = \alpha\sigma$, $\eta' = (\eta-1)/\alpha + 1$, $\lambda' = (\lambda-1)/\alpha + 1$.

For $\lambda > 1$ the distribution $P_0(x)$ diverges as $x \to 0$, therefore the diffusion of stochastic variable x should be restricted at least from the side of small values, or eq. (8) should be modified. The simplest choice of the restriction is the reflective boundary conditions at $x = x_{\mathrm{min}}$ and $x = x_{\mathrm{max}}$. However, other forms of restrictions are possible and have been considered, as well. Exponentially restricted diffusion is generated by the SDE

$$\begin{aligned}\mathrm{d}x \;=\; &\sigma^2\left[\eta - \frac{1}{2}\lambda + \frac{m}{2}\left(\frac{x_{\mathrm{min}}^m}{x^m} - \frac{x^m}{x_{\mathrm{max}}^m}\right)\right] x^{2\eta-1}\mathrm{d}t \\ &+\sigma x^\eta \mathrm{d}W\end{aligned} \tag{12}$$

obtained from eq. (8) by introducing the additional terms.

For $\lambda = 3$ we get that $\beta = 1$ and SDE (8) gives signal exhibiting $1/f$ noise. Numerical solution of the equations confirms the presence of the frequency region for which the power spectral density has $1/f^\beta$ dependence. The width of this region can be increased by increasing the ratio between the minimum and the maximum values of the stochastic variable x. In addition, the region in the power spectral density with the power-law behavior depends on the exponent η: if $\eta = 1$ then this width is zero; the width increases with increasing the difference $|\eta - 1|$ [42].

For some choices of parameters, SDE (8) or its variant (12) takes the form of the well-known SDEs considered in econopysics and finance. In case when the exponent of multiplicative noise $\eta = 0$ and $\sigma = 1$, (8) takes the form of the SDE for the *Bessel process* [44],

$$\mathrm{d}x = \frac{\delta-1}{2}\frac{1}{x}\mathrm{d}t + \mathrm{d}W, \tag{13}$$

of dimension $\delta = 1 - \lambda$, while $\eta = 1/2$ and $\sigma = 2$ corresponds to the *squared Bessel process* [44],

$$\mathrm{d}x = \delta\mathrm{d}t + 2\sqrt{x}\mathrm{d}W, \tag{14}$$

of dimension $\delta = 2(1-\lambda)$. SDE with exponential restriction (12) for $\eta = 1/2$, $x_{\mathrm{min}} = 0$ and $m = 1$ gives the *Cox-Ingersoll-Ross* (CIR) process [44],

$$\mathrm{d}x = k(\theta - x)\mathrm{d}t + \sigma\sqrt{x}\,\mathrm{d}W, \tag{15}$$

where $k = \sigma^2/2x_{\mathrm{max}}$ and $\theta = x_{\mathrm{max}}(1-\lambda)$. When $\nu = 2\eta$, $x_{\mathrm{max}} = \infty$ and $m = 2\eta - 2$, then eq. (12) takes the form of the *Constant Elasticity of Variance* (CEV) process [44],

$$\mathrm{d}x = \mu x\mathrm{d}t + \sigma x^\eta \mathrm{d}W, \tag{16}$$

where $\mu = \sigma^2(\eta-1)x_{\mathrm{min}}^{2(\eta-1)}$.

The numerical analysis of the proposed SDE (8) reveals the secondary structure of the signal composed of peaks or bursts, corresponding to the large deviations of the variable x from the proper average fluctuations. Bursts are characterized by power-law distributions of burst size, burst duration, and interburst time [31]. Therefore, proposed non-linear SDEs may simulate avalanches in self-organized critical (SOC) models and extreme event return times in long memory processes.

Herding model and $1/f$ noise. – Let us consider the case when $x \ll 1$. Then SDE (6) approximately has the form

$$\mathrm{d}x \approx h(\varepsilon_1 - \varepsilon_2 x)\mathrm{d}t + \sqrt{2hx}\mathrm{d}W. \tag{17}$$

Equation (17) has the form of our non-linear SDE (8) with the multiplicative noise and having parameters $\eta = 1/2$, $\lambda = 1 - \varepsilon_1$. The term with ε_2 in eq. (17) gives restrictions at larger x. The possible values of the parameter ε_1 are restricted to $\varepsilon_1 > 0$ and this limits the possible values of the exponents λ and β. In particular, it is not possible to obtain $1/f$ noise with $\beta = 1$ for the herding dynamics of population x. Nevertheless, the non-linear form of eq. (17) allows to apply transformation (10) and gets a flexible choice of variables and corresponding exponents λ and β.

One possible choice is to consider $y = 1/x$, having $\alpha = -1$ in eq. (10). The range of possible values of $1/x$ is $[1, +\infty)$. Since only large values of y are relevant for obtaining $1/f$ noise, this range can be extended to include zero by introducing $y = 1/x - 1 = (1-x)/x$. This variable y has a clear interpretation: it is equal to the ratio of the number of agents in the state 2 to the number of agents in the state 1:

$$y = \frac{1-x}{x} = \frac{N-n}{n}. \tag{18}$$

A stochastic variable similar to y (18) was used in ref. [9] to model absolute return, while our variable x corresponds to a fraction of fundamentalists in ref. [9]. Transformation (18) of variables leads from SDE (6) to

$$\mathrm{d}y = h[(2-\varepsilon_1)y + \varepsilon_2](1+y)\mathrm{d}t + \sqrt{2hy}(1+y)\mathrm{d}W. \tag{19}$$

A similar equation has been obtained in ref. [45]. The steady-state PDF of the new variable y is

$$P_0(y) = \frac{\Gamma(\varepsilon_1+\varepsilon_2)}{\Gamma(\varepsilon_2)\Gamma(\varepsilon_1)} \frac{y^{\varepsilon_2 - 1}}{(1+y)^{\varepsilon_2+\varepsilon_1}}. \tag{20}$$

When $y \gg 1$, then eq. (19) approximately has the form

$$\mathrm{d}y \approx h(2-\varepsilon_1)y^2\mathrm{d}t + \sqrt{2h}y^{\frac{3}{2}}\mathrm{d}W. \tag{21}$$

Equation (21) has the form of our non-linear SDE (8) with multiplicative noise, having parameters $\eta = 3/2$, $\lambda = 1 + \varepsilon_1$. Equation (21) is similar to a well-known 3/2 model of stochastic volatility [46]. According to eq. (9), the power exponent of the power spectral density is $\beta = \varepsilon_1 - 1$. If $\varepsilon_1 = 2$, we obtain the $1/f$ spectrum. Thus it is possible to obtain a signal with $1/f$ noise starting form the herding model. SDE (19) demonstrates yet another possible form of the restriction of diffusion of the stochastic variable in SDE (8)

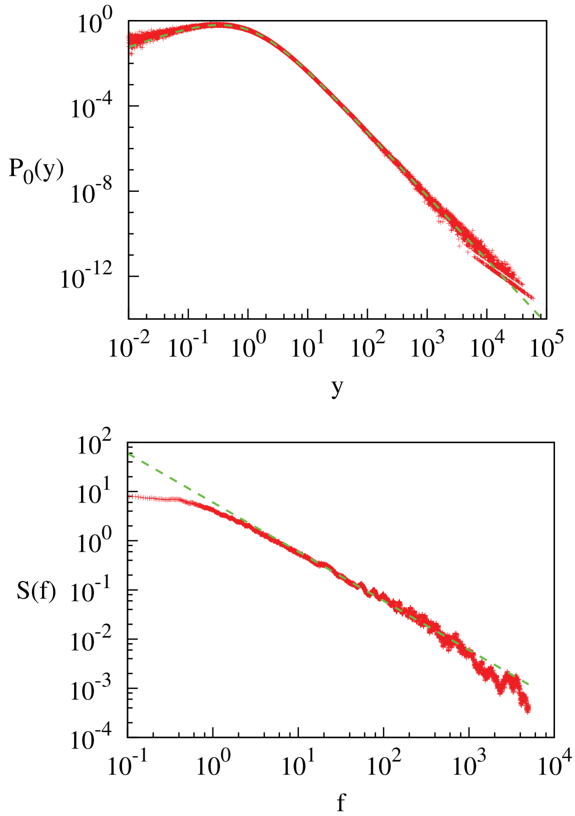

Fig. 1: (Color online) Steady-state PDF $P_0(y)$ (upper panel) and power spectral density $S(f)$ (lower panel) of the signal generated by eq. (19). The dashed (green) lines are the analytical expression (20) for the steady-state PDF (upper panel) and the slope $1/f$ (lower panel). The parameters used are $\varepsilon_1 = \varepsilon_2 = 2$ and $h = 1$.

from the side of small values. It is of interest to note, that the strong herding tendency for $h > \sigma_1/2$ ($\varepsilon_1 < 2$) yields the long-range process with $\beta < 1$, the power-law correlation $C(t) \sim 1/t^{1-\beta}$, and distribution (20) with the diverging variance.

The comparison of the numerically obtained steady-state PDF and the power spectral density of the signal generated by eq. (19) with analytical expressions is presented in fig. 1. For the numerical solution we use the Euler-Marujama approximation with variable step of integration, transforming the differential equations to the difference equations [32,33]. We see a good agreement of the numerical results with the analytical expressions. Numerical solution of the equations confirms the presence of the frequency region for which the power spectral density has $1/f$ dependence.

For the comparison, the steady-state PDF and the power spectral density of the stochastic variable y calculated using the number of agents n according to eq. (18), is presented if fig. 2. The number of agents n is obtained from the jump process defined by the transition probabilities (1) and (2). We see a good agreement of the numerical results with the analytical expressions. Thus, numerical

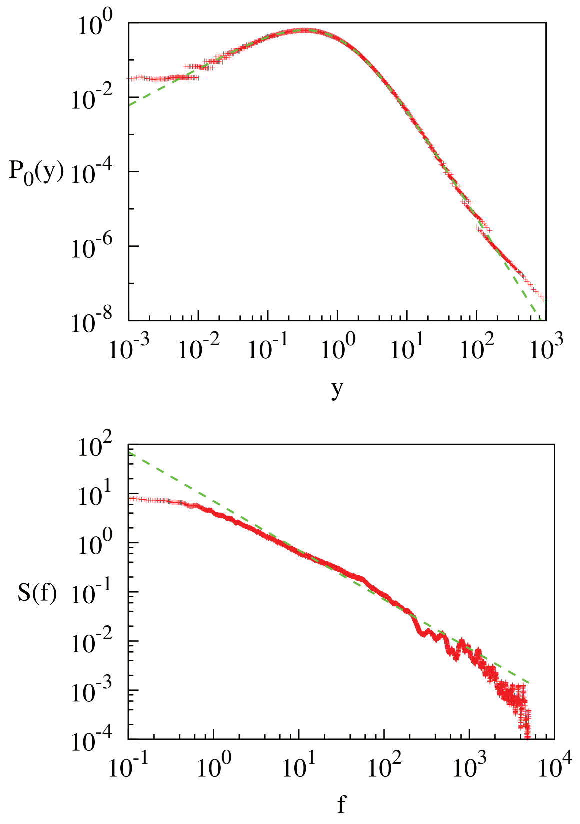

Fig. 2: (Color online) Steady-state PDF $P_0(y)$ (upper panel) and power spectral density $S(f)$ (lower panel) of the stochastic variable y calculated according to eq. (18), using the number of agents n obtained from the jump process defined by the transition probabilities (1) and (2). The dashed (green) lines are the analytical expression (20) for the steady-state PDF (upper panel) and the slope $1/f$ (lower panel). The total number of agents is $N = 10000$, other parameters are the same as in fig. 1.

calculations confirm the SDE (19) and the possibility to obtain the $1/f$ spectrum from a herding model.

Possible generalizations. – One of the possible generalizations of the model presented above is to consider a stochastic variable y defined as

$$y = \left(\frac{1-x}{x}\right)^{1/\alpha}. \tag{22}$$

Then the transformation of variables leads from SDE (6) to the SDE

$$\mathrm{d}y = \frac{h}{\alpha}\left[\left(1+\tfrac{1}{\alpha}-\varepsilon_1\right)+\left(\varepsilon_2+\tfrac{1}{\alpha}-1\right)y^{-\alpha}\right]y(1+y^{\alpha})\mathrm{d}t + \frac{\sqrt{2h}}{\alpha}y^{1-\frac{\alpha}{2}}(1+y^{\alpha})\mathrm{d}W. \tag{23}$$

The corresponding steady-state PDF is

$$P_0(y) = \frac{\alpha\Gamma(\varepsilon_1+\varepsilon_2)}{\Gamma(\varepsilon_2)\Gamma(\varepsilon_1)}\frac{y^{\alpha\varepsilon_2-1}}{(1+y^{\alpha})^{\varepsilon_2+\varepsilon_1}}. \tag{24}$$

It should be noted that PDF (24) for some choice of parameters has the form of distributions featured in non-extensive statistical mechanics [47–53]: the values of the parameters $\alpha = 1$, $\varepsilon_2 = 1$ correspond to a q-exponential distribution with $q = 1+1/(1+\varepsilon_1)$ and the values of the parameters $\alpha = 2$, $\varepsilon_2 = 1/2$ correspond to a q-Gaussian distribution with $q = 1+2/(1+2\varepsilon_1)$. When $y \gg 1$, then we get SDE (8) with parameters $\eta = 1+\alpha/2$, $\lambda = 1+\alpha\varepsilon_1$. According to eq. (9), the power exponent of the power spectral density is $\beta = \varepsilon_1 + 1 - 2/\alpha$.

Another way to generalize the herding model is to introduce the additional non-linearities into transition probabilities (1) and (2). The original transition probabilities (1) and (2) assume that agents meet at a constant rate and therefore the coefficients σ_1, σ_2 and h are constant. One possibility to extend the model is to assume that the rate at which the agents meet depends on the global state of the system. In such a situation the new transition probabilities can be written as

$$p(n \to n+1) = \frac{1}{\tau(n)}(N-n)(\sigma_1 + hn), \tag{25}$$

$$p(n \to n-1) = \frac{1}{\tau(n)}n(\sigma_2 + h(N-n)), \tag{26}$$

where $\tau(n)$ describes the time scale of the microscopic events. Assuming that the time scale τ depends only on the ratio $x = n/N$, the SDE obtained form the modified model instead of eq. (6) has the form

$$\mathrm{d}x = \frac{h}{\tau(x)}(\varepsilon_1(1-x)-\varepsilon_2 x)\mathrm{d}t + \sqrt{2\frac{h}{\tau(x)}x(1-x)}\mathrm{d}W, \tag{27}$$

whereas the SDE for the variable $y = (1-x)/x$ is

$$\mathrm{d}y = \frac{h}{\tau_y(y)}[(2-\varepsilon_1)y+\varepsilon_2](1+y)\mathrm{d}t + \sqrt{2\frac{h}{\tau_y(y)}y(1+y)}\mathrm{d}W. \tag{28}$$

Here $\tau_y(y) \equiv \tau(1/(1+y))$. A similar modification of the herding model has been proposed in ref. [45].

Let us consider the case of $\tau(y) = y^{-\gamma}$. Then eq. (28) becomes

$$\mathrm{d}y = h[(2-\varepsilon_1)y+\varepsilon_2]y^{\gamma}(1+y)\mathrm{d}t + \sqrt{2hy^{1+\gamma}}(1+y)\mathrm{d}W. \tag{29}$$

In the limit $y \gg 1$ we get SDE (8) with parameters $\eta = 3/2+\gamma/2$, $\lambda = \varepsilon_1+1+\gamma$. According to eq. (9), the power exponent of the power spectral density is

$$\beta = 1 + \frac{\lambda-3}{2(\eta-1)} = 1 + \frac{\varepsilon_1+\gamma-2}{1+\gamma}. \tag{30}$$

Thus we have shown that it is possible to obtain different values of the parameter η in eq. (8).

Conclusions. – Starting from a simple agent-based model describing herding behavior we obtained a non-linear SDE for the agent population x equal to the fraction of agents in the state 1. For $x \ll 1$ this non-linear SDE has a form similar to the SDE proposed in refs. [32,33] for the modeling of $1/f$ noise. This form suggests that it might be possible with the appropriate transformation of variables to obtain signals having $1/f$ behavior of the power spectral density from this agent model. However, the possible values of the parameter ε_1 in the model, eq. (17), are restricted to the positive values and this limits the values of the exponents λ and β of the power-law statistics. The solution is to consider not the fraction of agents x, but another variable y, equal to the ratio of number of agents in the state 2 to the number of agents in the state 1. This new variable is related to x by a simple transformation. The non-linear SDE for the stochastic variable y in the limit of large values $y \gg 1$ has the required form for obtaining the $1/f$ noise. This result shows that it is possible to obtain the $1/f$ noise starting from the agent-based herding model and introducing the appropriate variables. These analytical results are checked by numerical calculations, showing good agreement with analytical predictions. Thus, herding dynamics can be seen as a microscopic explanation of proposed non-linear SDEs generating signals with $1/f^\beta$ spectrum. The derivation of SDEs shows that non-linear terms in the transition probabilities, describing global interactions between agents, are crucial to the appearance of $1/f$ noise. The exponents λ and β of the power-law statistics can be adjusted by introducing feedback between the macroscopic system state x and the rate of the microscopic events $1/\tau(x)$. Application of a similar model for the description of the return in the finacial markets has been proposed in ref. [45].

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