Modeling scaled processes and clustering of events by the nonlinear stochastic differential equations

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Abstract. We present and analyze the nonlinear stochastic differential equations generating scaled signals with the power-law statistics, including $1/f^{\beta}$ noise and q-Gaussian distribution. Numerical analysis reveals that the process exhibits some peaks, bursts or extreme events, characterized by power-law distributions of the burst statistics and, therefore, the model may simulate self-organized critical and other systems exhibiting avalanches, bursts or clustering of events.

Keywords: 1/f noise, stochastic differential equations, q-Gaussian distribution
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INTRODUCTION

Power-law distributions, including $1/f$ noise, are ubiquitous in physics and in many other fields [1, 2, 3]. Despite the numerous models and theories, the intrinsic origin of 1/f noise and other scaled distributions still remain open questions. Starting from the multiplicative point process [4] we obtained the stochastic nonlinear differential equations, which generated signals with the power-law statistics, including $1/f^{\beta}$ fluctuations [3, 5]. Here the other nonlinear stochastic differential equation generating q-Gaussian distribution of the bursting signal and $1/f^{\beta}$ noise is presented and analyzed.

THE THEORY

We consider a nonlinear stochastic differential equation

$$\mathrm{d}x = \left(\eta - \frac{1}{2}\lambda\right)\left(x_m^2 + x^2\right)^{\eta-1} x\,\mathrm{d}t + \left(x_m^2 + x^2\right)^{\eta/2}\mathrm{d}W, \quad \eta > 1, \quad \lambda > 1 \tag{1}$$

generating q-Gaussian distributed signal

$$P(x) = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\lambda-1}{2}\right)x_m}\left(\frac{x_m^2}{x_m^2 + x^2}\right)^{\lambda/2} = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\lambda-1}{2}\right)x_m}\exp_q\left\{-\lambda\frac{x^2}{2x_m^2}\right\} \tag{2}$$

with $q = 1 + 2/\lambda$. Here W is a standard Wiener process and x_m is the parameter of the q-Gaussian distribution. Eq. (1) for small $x \ll x_m$ represents the linear additive stochastic

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process generating the Brownian motion with the linear relaxation, whereas for $x \gg x_m$ Eq. (1) reduces to the nonlinear multiplicative equation.

In accordance with Refs. [3, 4] the power spectrum of the process generated by Eq. (2) may be approximated as

$$S(f) = \frac{A}{(f_0^2 + f^2)^{\beta/2}} \tag{3}$$

with A characterizing the intensity of $1/f^{\beta}$ noise, $f_0 \sim f_{min}$ being the frequency for transition of spectrum at low frequencies to the flat spectrum, and

$$\beta = 1 + \frac{\lambda - 3}{2(\eta - 1)}. \tag{4}$$

The autocorrelation function of the process is

$$C(s) = \int_0^{\infty} S(f)\cos(2\pi f s)\mathrm{d}f = \frac{A\sqrt{\pi}}{\Gamma(\beta/2)}\left(\frac{\pi s}{f_0}\right)^h K_h(2\pi f_0 s), \tag{5}$$

with $K_h(z)$ being the modified Bessel function and $h = (\beta - 1)/2$. The second order structural function $F_2(s)$ and height-height correlation function $F(s)$ are expressed as

$$F(s) = F_2^2(s) = \left\langle |x(t+s) - x(t)|^2 \right\rangle = 2[C(0) - C(s)] = 4\int_0^{\infty} S(f)\sin^2(\pi s f)\mathrm{d}f. \tag{6}$$

Particular cases of Eqs. (5) and (6) are presented in Ref. [3].

NUMERICAL ANALYSIS

We present here the investigation results of the dependence of characteristics of Eq. (1) solutions on the nonlinearity parameter η for the fixed parameter $\lambda = 3$, i.e., for the pure $1/f$ noise.

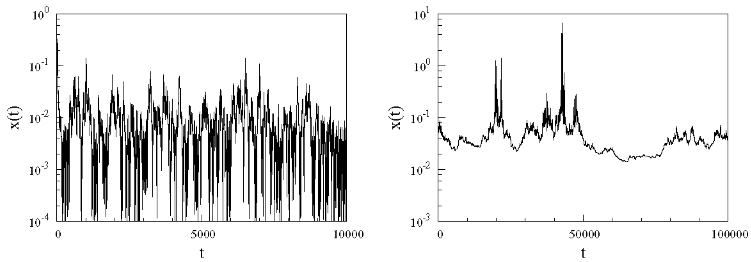

FIGURE 1. Examples of the numerically computed signals according to Eq. (1) with the parameters $\lambda = 3$, $x_m = 10^{-2}$, whereas $\eta = 1.5$ (left figure) and $\eta = 2.5$ (right figure).

As examples, in figure 1 we show the illustrations of the signals generated according to Eq. (1). We see bursts of the signal. In figures 2 and 3 the numerical calculations of the distribution density, $P(x)$, power spectral density, $S(f)$, autocorrelation function,

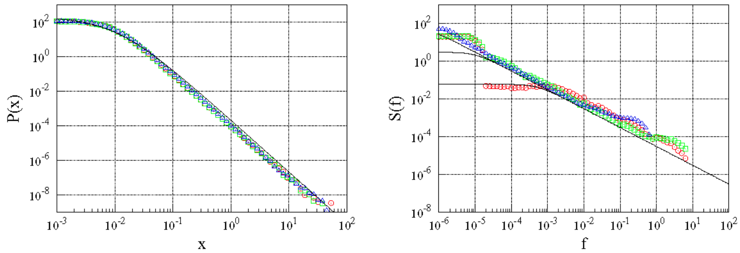

FIGURE 2. Distribution density, $P(x)$, and power spectral density, $S(f)$, for solutions of Eq. (1) with $\lambda = 3$, $x_m = 10^{-2}$ and different values of $\eta = 1.5$ (circles), $\eta = 2$ (squares) and $\eta = 2.5$ (triangles) in comparison with the analytical results (solid lines) according to Eqs. (2) and (3), respectively.

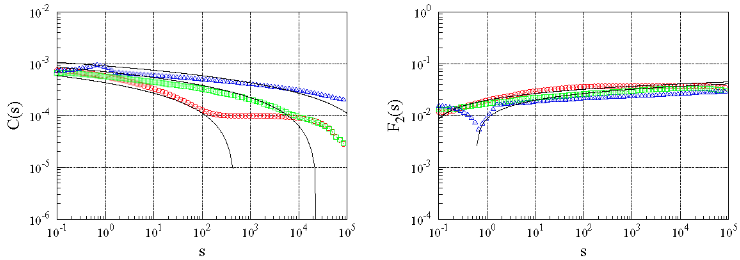

FIGURE 3. Autocorrelation function, $C(s)$, and the second order structural function, $F_2(s)$, for solutions of Eq. (1) with the same parameters as in figure 2 in comparison with the analytical results (solid lines) according to Eqs. (7) and (8), respectively.

$C(s)$, and the second order structural function, $F_2(s)$, for solutions of Eq. (1) with $\lambda = 3$, $x_m = 0.01$ and different values of the parameter η are presented. We see rather good agreement between the numerical calculations and the analytical results for $\beta = 1$,

$$C(s) = -A[\gamma + \ln(\pi f_0 s)] \tag{7}$$

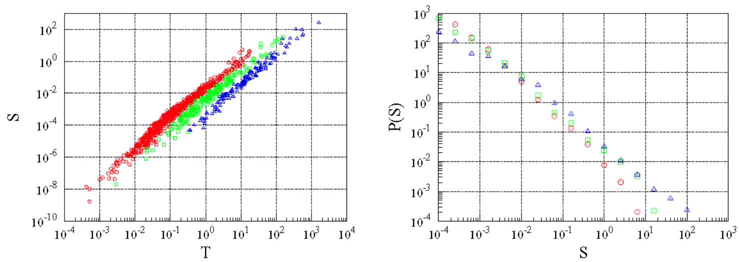

FIGURE 4. Dependence of the burst size S as a function of the burst duration T and distributions of the burst size, $P(S)$, for the peaks above the the threshold value $x_{th} = 0.1$. Calculations are as in figures 2 and 3 with the same parameters.

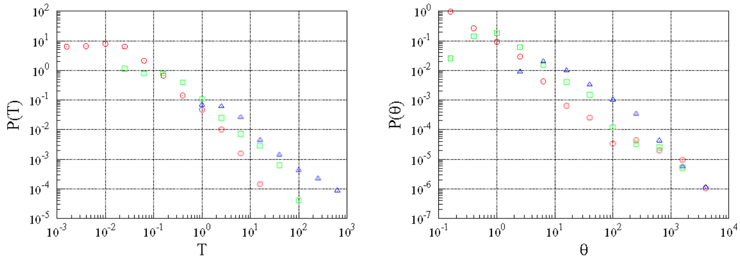

FIGURE 5. Burst duration, $P(T)$, and interburst time, $P(\theta)$, for the peaks above the the threshold value $x_{th} = 0.1$. Calculations are as in figure 4 with the same parameters.

$$F_2(s) = \sqrt{2A[\ln(\pi f_{\max} s) - \gamma]}, \tag{8}$$

where $\gamma = 0.577216$ is Euler's constant and f_{max} is the cutoff of the $1/f$ spectrum at high frequency. Figures 4 and 5 demonstrate that the size of the generated bursts S is approximately proportional to the squared burst duration T, i.e., $S \propto T^2$, and asymptotically power-law distributions of the burst size, $P(S) \sim S^{-1.3}$, burst duration, $P(T) \sim T^{-1.4}$ and interburst time, $P(\theta) \sim \theta^{-1.5}$, for the peaks above the threshold value x_{th} of the variable $x(t)$. These dependencies slightly depend on the degree of nonlinearity exponent η of the stochastic equation and are similar to those discovered [3] for the q-exponential distributions.

CONCLUSION

The nonlinear stochastic differential equations may generate q-Gaussian distributed signals with $1/f^{\beta}$ power spectrum, exhibiting bursts, similar to the crackling processes [6] and observable long-term memory time series [7, 8].

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