Nonlinear stochastic differential equation as the background of financial fluctuations

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Abstract. We present nonlinear stochastic differential equation (SDE) which forms the background for the stochastic modeling of return in the financial markets. SDE is obtained by the analogy with earlier proposed model of trading activity in the financial markets and generalized within the nonextensive statistical mechanics framework. Proposed stochastic model generates time series of return with two, the probability distribution function and the power spectral density, power-law statistics.

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INTRODUCTION

Empirical financial data exhibit the sophisticated and universal statistical properties. A variety of the so-called stylized facts has been established [1, 2], which can be seen as statistical signatures of the financial processes. The findings regarding the probability distribution function (PDF) of return and other financial variables are successfully generalized within the non-extensive statistical framework [3]. Additive-multiplicative stochastic models of the financial mean-reverting processes provide rich spectrum of shapes for PDF depending on the model parameters [4]. These models with appropriate fitting parameters do capture the distributions of returns, volatilities, and volumes but not necessarily the empirical temporal dynamics and correlations. There is empirical evidence that trading activity, trading volume, and volatility are stochastic variables with the long-range correlation [5, 6, 7] and this key aspect is not accounted for in the widespread models.

Recently we investigated the properties of stochastic multiplicative point processes analytically and numerically [8]. We derived formula for the power spectrum and related the model with the general form of multiplicative stochastic differential equations [9, 10]. Consequently, the stochastic model of trading activity based on the Poisson-like process driven by the nonlinear stochastic differential equation (SDE) was presented in Refs. [11, 12, 13, 14].

The statistical similarity of trading activity and absolute return together with the general background of non-extensive statistics give us an opportunity to model dynamics of return by nonlinear SDE. This forms a new theoretical approach of the stochastic modeling of financial variables.

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STOCHASTIC MODEL WITH Q-GAUSSIAN PDF AND LONG MEMORY

Starting from the multiplicative point process [8] we have derived SDE [9, 10]

$$\mathrm{d}x = \sigma^2 \left(\eta - \frac{\lambda}{2}\right) x^{2\eta-1}\mathrm{d}t + \sigma x^{\eta}\mathrm{d}W \tag{1}$$

which solution has power-law probability distribution function, $P(x) \sim x^{-\lambda}$, and power spectral density, $S(f) \sim 1/f^{\beta}$, in wide range of frequencies. Here

$$\beta = 1 - \frac{\lambda - 3}{2\eta - 2}, \qquad \frac{1}{2} < \beta < 2\,. \tag{2}$$

Due to the divergence of the power-law distribution and the requirement of the stationarity of the process, the stochastic equation (1) should be analyzed together with the appropriate restrictions of the diffusion in some finite interval $x_{\mathrm{min}} \leqslant x \leqslant x_{\mathrm{max}}$ [9].

Power spectral density is determined mainly by the power-law behavior of the coefficients of SDE (1) at big values of $x \gg x_{\mathrm{min}}$. Changing the coefficients at small x, the spectrum retains power-law behavior. Therefore, we propose the following modification of Eq. (1)

$$\mathrm{d}x = \sigma^2 \left(\eta - \frac{\lambda}{2}\right) (x_0^2 + x^2)^{\eta-1} x\mathrm{d}t + \sigma (x_0^2 + x^2)^{\eta/2}\mathrm{d}W. \tag{3}$$

The associated Fokker-Planck equation gives q-Gaussian PDF,

$$\begin{aligned} P(x) &= \frac{\Gamma(\lambda/2)}{\sqrt{\pi}x_0\Gamma((\lambda-1)/2)} \left(\frac{x_0^2}{x_0^2 + x^2}\right)^{\lambda/2} \\ &\equiv \frac{\Gamma(\lambda/2)}{\sqrt{\pi}x_0\Gamma((\lambda-1)/2)} \exp_q\left(-\lambda\frac{x^2}{2x_0^2}\right), \end{aligned}$$

with the parameter

$$q = 1 + 2/\lambda.$$

Here $\exp_q(\cdot)$ is q-exponential defined as

$$\exp_q(x) \equiv (1 + (1-q)x)^{1/(1-q)}. \tag{4}$$

Introducing scaled variables

$$\tilde{x} = x/x_0\,, \qquad \tilde{t} = \sigma^2 x_0^{2(\eta-1)} t$$

we get SDE

$$\mathrm{d}\tilde{x} = \left(\eta - \frac{\lambda}{2}\right)(1 + \tilde{x}^2)^{\eta-1}\tilde{x}\mathrm{d}\tilde{t} + (1 + \tilde{x}^2)^{\eta/2}\mathrm{d}\tilde{W} \tag{5}$$

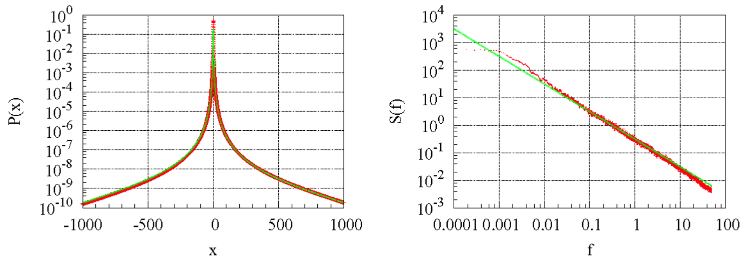

FIGURE 1. PDF, $P(x)$, of solution of Eq. (5) with parameters $\lambda = 3$ and $\eta = 5/2$ coinciding with the analytical Eq. (6) and spectrum, $S(f)$, in comparison with the line representing $1/f$ spectrum.

with stationary distribution

$$P(\tilde{x}) = \frac{\Gamma(\lambda/2)}{\sqrt{\pi}\Gamma((\lambda-1)/2)}\left(\frac{1}{1+\tilde{x}^2}\right)^{\lambda/2}. \tag{6}$$

In figure 1 the example solutions of equation (5) with parameters $\lambda = 3$ and $\eta = 5/2$ are presented.

EQUATION WITH TWO POWER-LAW EXPONENTS

In order to get spectrum with two power-law exponents we propose SDE

$$\mathrm{d}\tilde{x} = \left(\eta - \frac{\lambda}{2} - (\tilde{x}\varepsilon^{\eta})^2\right)\frac{(1+\tilde{x}^2)^{\eta-1}}{((1+\tilde{x}^2)^{1/2}\varepsilon+1)^2}\tilde{x}\mathrm{d}\tilde{t} + \frac{(1+\tilde{x}^2)^{\eta/2}}{(1+x^2)^{1/2}\varepsilon+1}\mathrm{d}\tilde{W}. \tag{7}$$

We solve Eq. (7) numerically using the method of discretization. Introducing a variable step of integration

$$h_k = \kappa^2\frac{((1+\tilde{x}_k^2)^{1/2}\varepsilon+1)^2}{(1+\tilde{x}_k^2)^{\eta-1}},$$

the differential equation (7) transforms to the difference equations

$$\begin{aligned} \tilde{x}_{k+1} &= \tilde{x}_k + \kappa^2\left(\eta - \frac{\lambda}{2} - (\tilde{x}_k\varepsilon^{\eta})^2\right)\tilde{x}_k + \kappa(1+\tilde{x}_k^2)^{1/2}\varepsilon_k, \\ t_{k+1} &= t_k + \kappa^2\frac{((1+\tilde{x}_k^2)^{1/2}\varepsilon+1)^2}{(1+\tilde{x}_k^2)^{\eta-1}}. \end{aligned}$$

We demonstrate an example solutions of equation (7) in figure 2 with parameters $\eta = 5/2$, $\lambda = 4.0$, and $\varepsilon = 0.01$. The numerical PDF fits very well the empirical histogram of ABT stocks traded on NYSE. Model recovers fractured behavior of absolute return power spectrum.

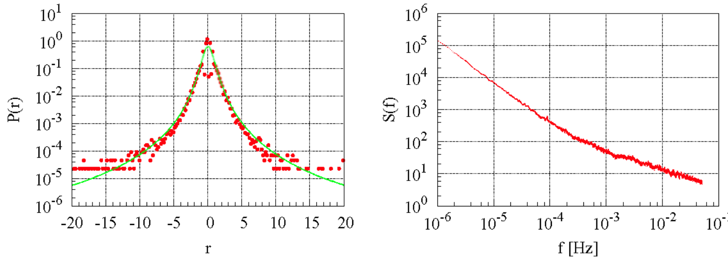

FIGURE 2. Model calculated from Eq. (7) with the parameters $\eta = 5/2$, $\lambda = 4.0$, and $\varepsilon = 0.01$ PDF, $P(x)$, (continuum line) in comparison with empirical histogram of one minute returns of ABT stocks traded on NYSE (dots) and power spectrum, $S(x)$, of returns.

CONCLUSION

We propose the nonlinear SDE reproducing the fascinating statistical properties of the financial variables with q-Gaussian PDF and fractured behavior of the power spectrum. The proposed stochastic model with empirically defined parameters reproduces the distribution of return and the correlations evaluated through the power spectral density of absolute return. Stochastic modeling of the financial variables by nonlinear SDE is consistent with the nonextensive statistical mechanics and provides new opportunities to capture empirical statistics in detail.

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