Simulation of bursting, rare and extreme events by nonlinear stochastic differential equations

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Abstract

We analyse the scaling properties of the signals generated by a nonlinear stochastic differential equation. The signals demonstrate the power-law statistics, including $1/f^{\beta}$ noise and q-Gaussian distribution. Numerical analysis is extended to a negative nonlinearity parameter and reveals that the generated process exhibits peaks, bursts or extreme events, characterized by power-law distributions of the burst statistics, as in a case of the positive parameter of the nonlinearity. Therefore, the model may simulate self-organized criticality (SOC) and other systems where the process consists of avalanches, bursts or clustering of the extreme events.

[1]

1 Introduction

In many systems one can often observe signals exhibiting long-term correlations characterized by a power-law decay of the autocorrelation function and with $1/f$ power spectra (see references in [1, 2, 3]). $1/f^{\beta}$ noise is observable in various systems with bursty or avalanche dynamics, as well. Despite the numerous models and theories, the intrinsic origin of 1/f noise and other scaled distributions still remain open questions. Most of the models and theories have restricted validity because of assumptions specific to the problem under consideration. A short categorization of the theories and models of $1/f$ noise is presented in the introduction of paper [3].

Recently, starting from the multiplicative point process [4] we obtained the stochastic nonlinear differential equations, which generated signals with the power-law statistics, including $1/f^{\beta}$ fluctuations [3, 5, 6]. The numerical analysis of the equations reveals the secondary structure of the signal composed of

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peaks, bursts, clusters of the events, corresponding to the large deviations of the variable x from the proper average fluctuations and with the power-law distributed burst sizes S, burst durations T, and the inter-burst time θ. Therefore, the proposed nonlinear stochastic model may simulate self-organized criticality (SOC) and other similar systems where the processes consist of avalanches, bursts or clustering of the extreme events [7, 8, 9, 10, 11, 12].

Here the another nonlinear stochastic differential equation generating q-Gaussian distribution of the bursting signal and $1/f^{\beta}$ noise is presented and analyzed.

2 The theory

Here we will consider a nonlinear stochastic differential equation

$$\mathrm{d}x = -\left(\frac{1}{2}\lambda - \eta\right)\left(x_m^2 + x^2\right)^{\eta-1} x\mathrm{d}t + \left(x_m^2 + x^2\right)^{\eta/2}\mathrm{d}W, \quad \lambda > 1 \tag{1}$$

generating q-Gaussian distributed signal

$$P(x) = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\lambda-1}{2}\right)x_m}\left(\frac{x_m^2}{x_m^2 + x^2}\right)^{\lambda/2} = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\lambda-1}{2}\right)x_m}\exp_q\left\{-\lambda\frac{x^2}{2x_m^2}\right\} \tag{2}$$

with $q = 1 + 2/\lambda$. Here W is a standard Wiener process and x_m is the parameter of the q-Gaussian distribution. Eq. (1) for small $x \ll x_m$ represents the linear additive stochastic process generating the Brownian motion with the linear relaxation, whereas for $x \gg x_m$ Eq. (1) reduces to the nonlinear multiplicative equation. In Ref. [13, 14] such equation has been analysed for the parameter $\eta > 1$. Recent theoretical analysis [15] revealed, however, that Eq. (1) exhibits $1/f^{\beta}$ noise for $\eta < 1$, as well. Here we analyse statistical properties of signals, generated by Eq. (1) with $\eta = 0$ and $\eta = -1/2$.

In accordance with Refs. [3, 4] the power spectrum of the process generated by Eq. (2) may be approximated as

$$S(f) = \frac{A}{(f_0^2 + f^2)^{\beta/2}} \tag{3}$$

with A characterizing the intensity of $1/f^{\beta}$ noise, $f_0 \sim f_{min}$ being the frequency for transition at low frequencies to the flat spectrum, and

$$\beta = 1 + \frac{\lambda - 3}{2(\eta - 1)}. \tag{4}$$

The autocorrelation function of the process is

$$C(s) = \int_0^{\infty} S(f)\cos(2\pi f s)\mathrm{d}f = \frac{A\sqrt{\pi}}{\Gamma(\beta/2)}\left(\frac{\pi s}{f_0}\right)^h K_h(2\pi f_0 s), \tag{5}$$

with $K_h(z)$ being the modified Bessel function and $h = (\beta - 1)/2$. The second order structural function $F_2(s)$ and height-height correlation function $F(s)$ are expressed as

$$F(s) = F_2^2(s) = \left\langle |x(t+s) - x(t)|^2 \right\rangle = 2[C(0) - C(s)] = 4\int_0^\infty S(f)\sin^2(\pi s f)\mathrm{d}f. \quad (6)$$

Particular cases of Eqs. (5) and (6) are presented in Ref. [3].

3 Numerical analysis

We present the results of the numerical investigation of the dependence of characteristics of Eq. (1) solutions on the nonlinearity parameter $\eta < 1$ for the fixed parameter $\lambda = 3$, i.e., for the pure $1/f$ noise. Thus for $\eta = 0$ and $\eta = -1/2$ Eq. (1) takes the form

$$\mathrm{d}x = -\frac{3}{2}\frac{x}{x_m^2 + x^2}\mathrm{d}t + \mathrm{d}W, \quad \eta = 0 \quad (7)$$

and

$$\mathrm{d}x = -2\left(x_m^2 + x^2\right)^{-3/2} x\mathrm{d}t + \left(x_m^2 + x^2\right)^{-1/4}\mathrm{d}W, \quad \eta = -1/2, \quad (8)$$

respectively.

Eqs. (7) and (8) can be solved using the method of discretization with the constant step of integration, $\Delta t \ll 1$,

$$x_{k+1} = x_k - \frac{3}{2}\frac{x_k}{x_m^2 + x_k^2}\Delta t + \sqrt{\Delta t}\varepsilon_k, \quad \eta = 0, \quad (9)$$

$$x_{k+1} = x_k - 2\left(x_m^2 + x_k^2\right)^{-3/2} x_k\Delta t + \left(x_m^2 + x_k^2\right)^{-1/4}\sqrt{\Delta t}\varepsilon_k, \quad \eta = -1/2, \quad (10)$$

or using the variable step of integration,

$$\Delta t_k = \kappa^2\left(x_m^2 + x^2\right)^{1-\eta}, \quad (11)$$

which results in the universal difference equation

$$x_{k+1} = x_k - \left(\frac{1}{2}\lambda - \eta\right)\kappa^2 x_k + \kappa\sqrt{x_m^2 + x_k^2}\varepsilon_k. \quad (12)$$

Here ε_i is a set of uncorrelated normally distributed random variables with zero expectation and unit variance and κ is a small parameter. Eq. (12) corresponds the case when the change of the variable x in one step is proportional to the value of the variable at time of the step.

As examples, in figure 1 we show the illustrations of the signals generated according to Eqs. (9) and (10). We see bursting signals, similar to the observed for the exponent $\eta > 1$ [3, 13]. In figures 2 and 3 the numerical calculations of the distribution density, $P(x)$, power spectral density, $S(f)$, autocorrelation

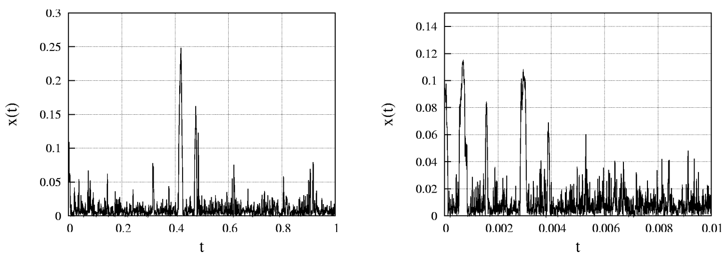

Figure 1: Examples of the numerically computed signals according to Eq. (7) (left figure) and Eq. (8) (right figure) with the parameter $x_m = 10^{-2}$.

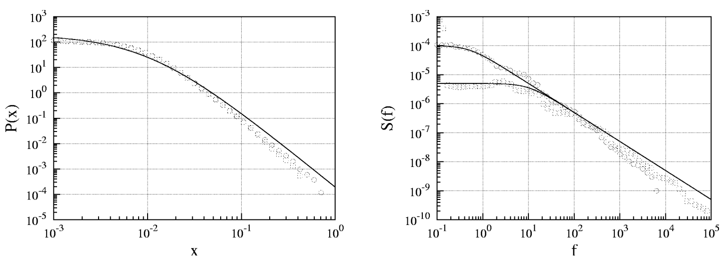

Figure 2: Distribution density, $P(x)$, and power spectral density, $S(f)$, for solutions of Eqs. (11) and (12) with $\lambda = 3$, $x_m = 10^{-2}$ and different values of $\eta = 0$ (circles), $\eta = -0.5$ (squares) in comparison with the analytical results (solid lines) according to Eqs. (2) and (3), respectively.

function, $C(s)$, and the second order structural function, $F_2(s)$, for solutions of Eq. (1) with $\lambda = 3$, $x_m = 0.01$ and different values of the parameter η are presented. We see agreement between the numerical calculations and the analytical results [3] for $\beta = 1$,

$$C(s) = -A[\gamma + \ln(\pi f_0 s)] \tag{13}$$

$$F_2(s) = \sqrt{2A[\ln(\pi f_{\max} s) - \gamma]}, \tag{14}$$

where $\gamma = 0.577216$ is Euler's constant and f_{max} is the cutoff of the $1/f$ spectrum at high frequency. Some deviation of the simulated results for the distribution density $P(x)$ in Fig. 2 from the asymptotic of the analytical distribution is owing to the technical reason, i.e., due to the finite time of calculation it is difficult to observe the large values of the variable x with very small probability. This reason, as well as the deviation of the distribution from the power-law $1/x^3$ for small x, determines the deviation of the power spectral density $S(f)$ in Fig. 2, the autocorrelation function $C(s)$ and the second order structural function $F_2(s)$ in Fig. 3 from the analytical results, estimated for the ideal power-law distributions.

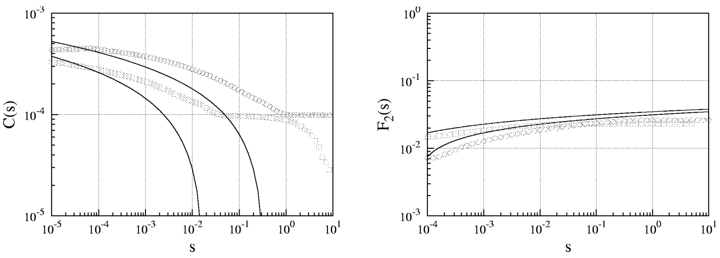

Figure 3: Autocorrelation function, $C(s)$, and the second order structural function, $F_2(s)$, for solutions of Eq. (1) with the same parameters as in figure 2 in comparison with the analytical results (solid lines) according to Eqs. (13) and (14), respectively.

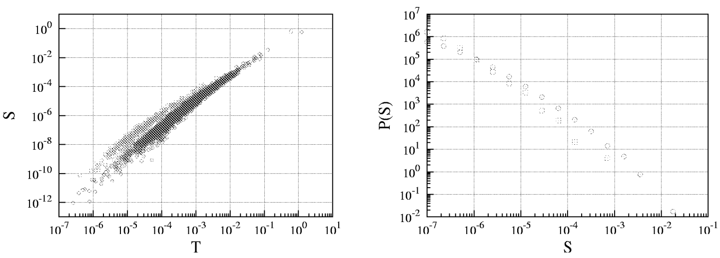

Figure 4: Dependence of the burst size S as a function of the burst duration T and distributions of the burst size, $P(S)$, for the peaks above the the threshold value $x_{th} = 0.1$. Calculations are with the same parameters as in figures 2 and 3.

Figures 4 and 5 demonstrate that the size of the generated bursts S is approximately proportional to the squared burst duration T, i.e., $S \propto T^2$, and asymptotically power-law distributions of the burst size, $P(S) \sim S^{-1.5}$, burst duration, $P(T) \sim T^{-2}$ and interburst time, $P(\theta) \sim \theta^{-1}$, for the peaks above the threshold value $x_{th} = 0.1$ of the variable $x(t)$. These dependencies slightly depend on the degree of nonlinearity exponent η of the stochastic equation.

4 Conclusion

The nonlinear stochastic differential equations with nonlinearity exponent $\eta < 1$ generate q-Gaussian distributed signals with $1/f^\beta$ power spectrum, exhibiting bursts with the power-law statistics, similar to those discovered for the q-exponential [3] and q-Gaussian [13] distributions for the nonlinearity exponent $\eta > 1$. The burst sizes are approximately proportional to the squared duration of the burst. On the other hand, the analysed model reproduces $1/f$ noise and the processes not only in SOC, crackling systems and observable long-term

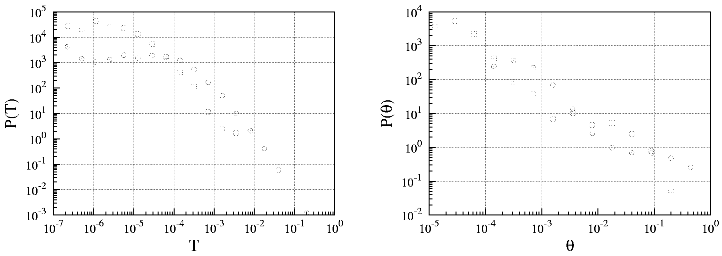

Figure 5: Burst duration, $P(T)$, and interburst time, $P(\theta)$, for the peaks above the the threshold value $x_{th} = 0.1$. Calculations are with the same parameters as in figure 4.

memory time series [7, 8, 9, 10, 11, 12, 16, 17, 18] but it is related with the clustering of events described by the driven Poisson process [14] and superstatistical approach [19], as well.

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