Scaling properties of signals as origin of $1/f$ noise

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Abstract. There are several mathematical models yielding $1/f$ noise. For example, a $1/f$ spectrum can be obtained from a stochastic sequence of pulses with a power-law distribution of pulse durations or from nonlinear stochastic differential equations. We show that a couple of seemingly different models exhibiting a $1/f$ spectrum are due to similar scaling properties of the signals. In addition, we demonstrate a connection between signals with the power-law behavior of the power spectral density generated by the nonlinear stochastic differential equations and modeled by a sequence of random different pulses. An approximation of solutions of the nonlinear stochastic differential equations by the sequence of pulses correctly reproduces the power-law parts of the probability density function and of the power spectral density. This connection provides further insights into the origin of $1/f$ noise.

Contents

1. Introduction

Signals with power spectral density (PSD) at low frequencies f of the form $S(f) \sim 1/f^\beta$ with β close to 1 are commonly referred to as '$1/f$ noise', '$1/f$ fluctuations', or 'flicker noise.' Power-law distributions of spectra of signals with $0.5 < \beta < 1.5$, as well as scaling behavior, are ubiquitous in physics and in many other fields [1–7]. Despite the numerous models and theories proposed since its discovery more than 80 years ago [8, 9], the subject of $1/f$ noise remains open for new discoveries. Most models and theories of $1/f$ noise are not universal because of the assumptions specific to the problem under consideration. A short categorization of the theories and models of $1/f$ noise is presented in the introduction of the paper [10]. See also recent review by Balandin [11].

Often $1/f$ noise is modeled as the superposition of Lorentzian spectra with a wide range distribution of relaxation times [12]. An influential class of the models of $1/f$ noise involves self-organized criticality (SOC). In 1987 Bak *et al* [13] introduced the notion of SOC with the motivation to explain the universality of $1/f$ noise. Although paper [13] is the most cited paper in the field of $1/f$ noise problems, it was shown later on [14, 15] that the mechanism proposed in [13] results in $1/f^\beta$ fluctuations with $1.5 < \beta \leqslant 2$. The $1/f$ noise in the fluctuations of mass was first seen in a sandpile model with threshold dissipation by Ali [16].

Other models of $1/f$ noise involve a class of maps generating intermittent signals. It is possible to generate power-laws and $1/f$ noise from simple iterative maps by fine-tuning the parameters of the system at the edge of chaos [17] where the sensitivity to initial conditions of the logistic map is a lot milder than in the chaotic regime [18]. Manneville [19] showed that an iterative function can produce interesting behavior, power-laws and $1/f$ PSD. In paper [20] a mechanism of intermittency exhibiting $1/f$ noise, which occurs in nonlinear dynamical systems with invariant subspace and having the transverse Lyapunov exponent equal to zero, is considered.

In many cases the physical processes can be represented by a sequence of random pulses. The mathematical way of generating power-law noise from a sequence of pulses has been discussed by Halford [21]. The spectrum of a signal consisting of

pulse sequences belonging to the Markov process, was investigated in [22, 23]. In those models the power-law PSD appears due to the power-law distribution of pulse durations. The main objection to this approach is an apparent lack of physical mechanism generating the durations over several orders of magnitude [24]. On the other hand, we can point out that the diffusion process gives rise to broad distributions of lifetimes. It is known that for the unbiased random walk, the distribution of the first return times has a power-law form with the exponent $-3/2$ [25]. Another possible mechanism leading to broad distribution of lifetimes is due to the formation of avalanches. In many situations (for example in Barkhausen noise [26], fluid invasion into disordered media [27], dislocation avalanches in microcrystals [28]) the volume of avalanches has power-law distribution and also the volume scales as a power-law function of the duration of the avalanche. Similar properties also have avalanches in the models of SOC [29–31].

One more way of obtaining $1/f$ noise from a signal consisting of pulses has been presented in [32–34]. It has been shown that the intrinsic origin of $1/f$ noise may be a Brownian motion of the time between each signal pulse, similar to the Brownian fluctuations of the signal amplitude, resulting in $1/f^2$ noise. Recently, the nonlinear stochastic differential equations (SDEs) generating signals with $1/f$ noise were obtained in [35, 36] starting from the point process model of $1/f$ noise. Analysis of the long-range correlated bursting signals is presented in [37–39], as well.

The purpose of this paper is to show the connection between the nonlinear SDEs generating signals with $1/f$ noise and signals consisting of random pulses with the power-law distribution of pulse durations. As we will demonstrate, in both of these models $1/f$ spectrum appears due to the scaling properties of the signal. In addition, the signal generated by SDEs can be approximated by rectangular pulses yielding the same distribution of signal intensity and the same power-law exponent in the PSD. Although the models generating $1/f$ noise that we consider in this paper (nonlinear SDEs and random pulses with the power-law distribution of pulse durations) are not unique, the method of their derivation from the scaling properties of the signal has not been investigated before. We obtain nonlinear SDEs generating signals with $1/f$ noise starting *not* from the point process model, as was done in [35, 36], but *from the scaling properties of the signal required to get $1/f$ noise*. This approach allows us to reveal new connections between those seemingly different models.

The paper is organized as follows: In section 2 we consider nonlinear SDEs generating signals with $1/f^\beta$ PSD and show that such SDEs can be obtained by requiring a proper scaling. In section 3 we analyze signals consisting of random pulses with the power-law distribution of pulse durations and power-law dependence of pulse height on the pulse duration. We show that such pulses have the same scaling properties as the signal generated by SDEs in section 2. In section 4 we produce the connection between the nonlinear SDEs' modeling and the rectangular pulses' series more explicitly. Section 5 summarizes our findings.

2. Nonlinear SDE generating signals with $1/f^\beta$ noise from scaling

Nonlinear SDEs generating signals with $1/f^\beta$ PSD are derived in papers [35, 36]. In this section we show these SDEs can be obtained only from the scaling properties required

by $1/f^\beta$ PSD. This new technique reveals more directly the origin of $1/f^\beta$ PSD compared to the derivation starting from the point process model, as was done in [35, 36].

Pure $1/f^\beta$ PSD is physically impossible because the total power would be infinite. Therefore, we will consider signals with PSD having $1/f^\beta$ behavior only in some wide intermediate region of frequencies, $f_{\min} \ll f \ll f_{\max}$, whereas for small frequencies $f \ll f_{\min}$ PSD is bounded. We can obtain nonlinear SDE generating signals exhibiting $1/f$ noise using the following considerations. Wiener–Khintchine theorem relates PSD $S(f)$ to the autocorrelation function $C(t)$:

$$C(t) = \int_0^{+\infty} S(f) \cos(2\pi f t) \, \mathrm{d}f. \tag{1}$$

If $S(f) \sim f^{-\beta}$ in a wide region of frequencies, then for the frequencies in this region the PSD has a scaling property

$$S(af) \sim a^{-\beta} S(f) \tag{2}$$

when the influence of the limiting frequencies $f_{\min}$ and $f_{\max}$ is neglected. From the Wiener–Khintchine theorem (1) it follows that the autocorrelation function has the scaling property

$$C(at) \sim a^{\beta-1} C(t) \tag{3}$$

in the time range $1/f_{\max} \ll t \ll 1/f_{\min}$. The autocorrelation function can be written as [40–42]

$$C(t) = \int \mathrm{d}x \int \mathrm{d}x' \, xx' P_0(x) P(x', t | x, 0), \tag{4}$$

where $P_0(x)$ is the steady state probability density function (PDF) and $P(x', t | x, 0)$ is the transition probability (the conditional probability that at time t the signal has value x' with the condition that at time $t = 0$ the signal had the value x). The transition probability can be obtained from the solution of the Fokker–Planck equation with the initial condition $P(x', t | x, 0) = \delta(x'-x)$. One of the ways to obtain the required property (3) is for the steady state PDF to have the power-law form

$$P_0(x) \sim x^{-\lambda} \tag{5}$$

and for the transition probability to have the scaling property

$$aP(ax', t | ax, 0) = P(x', a^\mu t | x, 0), \tag{6}$$

where μ is the scaling exponent, the meaning of which will be revealed below, equation (12). Indeed, from equations (4)–(6) and a change of variables it follows

$$C(at) = \int \mathrm{d}x \int \mathrm{d}x' \, xx' P_0(x) P(x', at | x, 0) \tag{7}$$

$$\sim \int \mathrm{d}x \int \mathrm{d}x' \, x^{1-\lambda} x' \, a^{\frac{1}{\mu}} P(a^{\frac{1}{\mu}} x', t | a^{\frac{1}{\mu}} x, 0) \tag{8}$$

$$\sim a^{\frac{\lambda-3}{\mu}} \int \mathrm{d}u \int \mathrm{d}u' \, uu' P_0(u) P(u', t | u, 0). \tag{9}$$

Thus, the autocorrelation function has the required property equation (3) with β given by equation

$$\beta = 1 + (\lambda - 3)/\mu. \tag{10}$$

Note that, according to equation (6), the change of the magnitude of the stochastic variable $x \to ax$ is equivalent to the change of time scale $t \to a^\mu t$.

In order to avoid the divergence of steady state PDF (5) the diffusion of stochastic variable x should be restricted at least from the side of small values and, therefore, equation (5) holds only in some region of the variable x, $x_{\min} \ll x \ll x_{\max}$. When the diffusion of stochastic variable x is restricted, equation (6) also cannot be exact. However, if the influence of the limiting values $x_{\min}$ and $x_{\max}$ can be neglected for time t in some region $t_{\min} \ll t \ll t_{\max}$, we can expect that the scaling equation (3) approximately holds in this time region.

To get the required scaling equation (6) of the transition probability, the SDE should contain only powers of the stochastic variable x. This will be the case if the coefficient in the noise term is the power-law depending, i.e., proportional to x^η. The drift term then is fixed by the requirement (5) for the steady state PDF. Thus we consider SDE [10]

$$\mathrm{d}x = \sigma^2 \left(\eta - \frac{1}{2}\lambda \right) x^{2\eta-1}\mathrm{d}t + \sigma x^\eta \mathrm{d}W_{\mathrm{t}}. \tag{11}$$

Here W_t is a standard Wiener process (the Brownian motion) and σ is the white noise intensity. Note that SDE equation (11) is the same as in papers [10, 36], only here we have obtained it from consideration of the scaling properties, not starting from the point process model. Changing the variable x in (11) to the scaled variable $x_{\mathrm{s}} = ax$ or introducing the scaled time $t_{\mathrm{s}} = a^{2(\eta-1)}t$ and using the property of the Wiener process $\mathrm{d}W_{t_{\mathrm{s}}} = a^{\eta-1}\mathrm{d}W_{\mathrm{t}}$ one gets the same resulting equation. Thus, change of the scale of the variable x and change of time scale are equivalent, as in equation (6) and the exponent μ is

$$\mu = 2(\eta - 1). \tag{12}$$

From equation (10) it follows that the power-law exponent in the PSD of the signal generated by SDE (11) is

$$\beta = 1 + \frac{\lambda - 3}{2(\eta - 1)}. \tag{13}$$

In order to obtain a stationary process and avoid divergence of steady state PDF the diffusion of stochastic variable x should be restricted or equation (11) should be modified. The simplest choice of the restriction is the reflective boundary conditions at $x = x_{\min}$ and $x = x_{\max}$. Another choice would be modification of equation (11) to get a rapidly decreasing steady state PDF when the stochastic variable x acquires values outside of the interval $[x_{\min}, x_{\max}]$. For example, the steady state PDF

$$P_0(x) \sim \frac{1}{x^\lambda}\exp\left\{ -\left(\frac{x_{\min}}{x}\right)^m - \left(\frac{x}{x_{\max}}\right)^m \right\} \tag{14}$$

with $m > 0$ has a power-law form when $x_{\min} \ll x \ll x_{\max}$ and exponential cut-offs when x is outside of the interval $[x_{\min}, x_{\max}]$. Such exponentially restricted diffusion is generated by the SDE

$$\mathrm{d}x = \sigma^2 \left[\eta - \frac{1}{2}\lambda + \frac{m}{2} \left(\frac{x_{\min}^m}{x^m} - \frac{x^m}{x_{\max}^m} \right) \right] x^{2\eta-1} \mathrm{d}t + \sigma x^\eta \mathrm{d}W_t \tag{15}$$

obtained from equation (11) by introducing additional terms in the drift.

The presence of the restrictions at $x = x_{\min}$ and $x = x_{\max}$ makes the scaling equation (6) not exact and this limits the power-law part of the PSD to a finite range of frequencies $f_{\min} \ll f \ll f_{\max}$. Let us estimate the limiting frequencies. Taking into account the limiting values $x_{\min}$ and $x_{\max}$, equation (6) for the transition probability corresponding to SDE (11) becomes

$$aP(ax', t| ax, 0; ax_{\min}, ax_{\max}) = P(x', a^\mu t| x, 0; x_{\min}, x_{\max}). \tag{16}$$

Here $x_{\min}$, $x_{\max}$ are the parameters of the transition probability. The steady state distribution $P_0(x; x_{\min}, x_{\max})$ has the scaling property

$$aP_0(ax; ax_{\min}, ax_{\max}) = P_0(x; x_{\min}, x_{\max}). \tag{17}$$

Inserting equations (16) and (17) into equation (4) we obtain

$$C(t; ax_{\min}, ax_{\max}) = a^2 C(a^\mu t, x_{\min}, x_{\max}). \tag{18}$$

This equation means that time t in the autocorrelation function should enter only in combinations with the limiting values, $x_{\min}t^{1/\mu}$ and $x_{\max}t^{1/\mu}$. We can expect that the influence of the limiting values can be neglected and the scaling equation (6) holds when the first combination is small and the second large; that is, when time t is in the interval $\sigma^{-2}x_{\max}^{-\mu} \ll t \ll \sigma^{-2}x_{\min}^{-\mu}$. Then, using equation (1) the frequency range where the PSD has $1/f^\beta$ behavior can be estimated as

$$\sigma^2 x_{\min}^\mu \ll 2\pi f \ll \sigma^2 x_{\max}^\mu. \tag{19}$$

However, numerical solutions of proposed nonlinear SDEs show that this estimation is too broad, i.e., the numerically obtained frequency region with the power-law behavior of PSD is narrower than according to equation (19). Note that for $\mu = 0$, i.e., $\eta = 1$ the width of the frequency region equation (19) is zero and we do not have $1/f^\beta$ power spectral density.

Comparison of the numerically obtained steady state PDF and the PSD with analytical expressions for SDE (11) with $\eta = \mu = 2$ and $\lambda = 3$ is presented in figure 1. For the numerical solution we use the Euler–Maruyama approximation, transforming the differential equations to difference equations. We can use a constant time step; however, at large values of x the coefficients in the equations become large and thus require a very small time step. A more effective solution is to use a variable time step, decreasing with the increase of x. As in [35, 36] we choose the time step in such a way that the coefficient before noise becomes proportional to the first power of x. One also gets very similar numerical results by using the Milstein approximation [10]. We see good agreement of the numerical results with the analytical expressions. A numerical solution of the equations confirms the presence of the frequency region for which the power spectral density has $1/f^\beta$ dependence. The $1/f$ interval in the PSD in figure 1 is
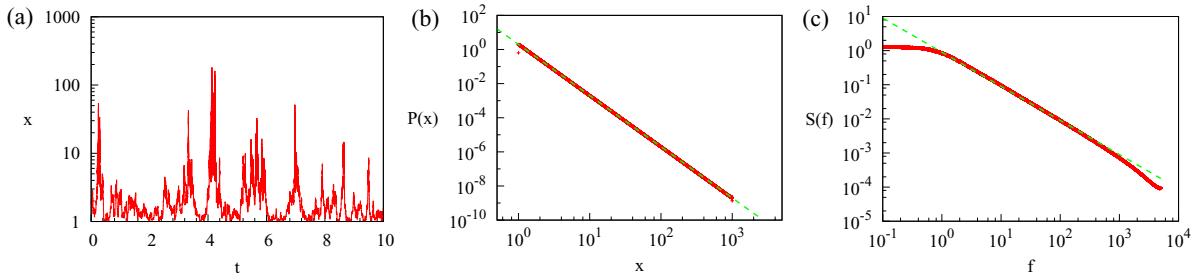
Figure 1. (*a*) Typical signal generated by equation (11) with reflective boundaries at $x_{\min}$ and $x_{\max}$. (*b*) The PDF of the signal intensity. The dashed (green) line shows the power-law with the exponent -3. (*c*) The PSD of such a signal. The dashed (green) line shows the slope f^{-1}. Used parameters are $\eta = 2$, $\lambda = 3$, $x_{\min} = 1$, $x_{\max} = 1000$ and $\sigma = 1$.

approximately between $f_{\min} \approx 10^0$ and $f_{\max} \approx 10^3$ and is much narrower than the width of the region $1 \ll f \ll 10^6$ predicted by equation (19). The width of this region can be increased by increasing the ratio between the minimum and the maximum values of the stochastic variable x.

As we see in figure 1(*a*), the numerical calculations exhibit a structure of the signal, consisting of peaks or bursts. Analysis [10] reveals that the sizes of the bursts are approximately proportional to the squared durations of the bursts with the power-law distributions of the bursts' durations and the interburst time. The exponent of the PDF of the interburst time approximately equal to $-3/2$ has been obtained numerically [10] and analytically [43].

3. Stochastic pulse sequences

In this section we consider pulse sequences with independent pulses. The shapes of the pulses are characterized by an arbitrary large set of parameters ξ, whereas the occurrence times of the pulses are described by a set of time moments $\{t_k\}$. The general form of the signal can be written as

$$I(t) = \sum_k A(t - t_k, \xi_k), \tag{20}$$

where functions $A(t, \xi)$ determine the shape of individual pulses. The pulse duration τ is included in the set of parameters ξ or, more generally, is a function of the parameters, $\tau(\xi)$. Inter-pulse duration is $\vartheta_k = t_{k+1} - t_k$. Such a pulse sequence is schematically shown in figure 2. We assume that: (i) the pulse sequences are stationary and ergodic; (ii) parameters ξ of different pulses are independent; (iii) all pulses are described by the same function $A(t, \xi)$; (iv) the pulse parameters ξ have the distribution $P(\xi)$.

The easiest way to calculate the PSD of a pulse sequence is to start from the PSD's definition directly [22]. However, in order to demonstrate a connection with the previous section we will consider the expression for the autocorrelation function of the signal. The autocorrelation function is given by the equation
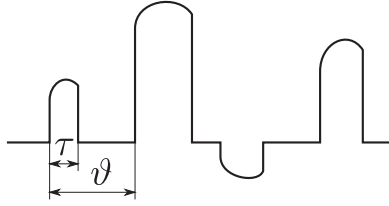
Figure 2. Sequence of random pulses.

$$C(t) = \lim_{T \to \infty} \left\langle \frac{1}{T-t} \int_0^{T-t} I(t')\,I(t'+t)\,\mathrm{d}t' \right\rangle, \tag{21}$$

where T the observation time interval and the brackets $\langle \bullet \rangle$ denote averaging over realizations of the pulse sequence. Using the signal (20) we can write

$$C(t) = \lim_{T \to \infty} \left\langle \frac{1}{T} \sum_{k,k'=1}^N \int_0^\infty A(t',\xi_k)\,A(t'+t+t_k-t_{k'},\xi_{k'})\,\mathrm{d}t' \right\rangle, \tag{22}$$

where N is the number of pulses during the observation time interval T.

The autocorrelation function can be decomposed into two parts, the first part containing the autocorrelation of each pulse with itself and the second part containing all cross terms:

$$C(t) = \lim_{T \to \infty} \left\langle \frac{1}{T} \sum_{k=1}^N \int_0^\infty A(t',\xi_k)\,A(t'+t,\xi_k)\,\mathrm{d}t' \right\rangle + \text{other terms}. \tag{23}$$

In many cases $1/f^\beta$ PSD is caused only by the first part. For example, it is known that when the pulses occur randomly as a Poisson process, the PSD of the signal depends only on the shapes of the pulses, as given by Carson's theorem [44]. Conditions when a sequence of randomly occurring pulses leads to $1/f^\beta$ noise were investigated in [21]. Note, that when inter-pulse duration $\vartheta_k = t_{k+1} - t_k$ is uncorrelated with the duration of the pulse, different pulses may be overlapping. Even in case when the other terms in equation (23) are nonzero, the first part can have different dependence on time t and dominate for some time range $t_{\min} < t < t_{\max}$. Therefore, initially we will consider only the first part of equation (23). It can be written as

$$C(t) = \nu \left\langle \int_0^\infty A(t',\xi)\,A(t'+t,\xi)\,\mathrm{d}t' \right\rangle, \tag{24}$$

where ν is the mean number of pulses per unit time. Since the pulse duration τ is a function of the parameters ξ, the PDF of pulse durations is

$$P_\tau(\tau') = \int \delta(\tau(\xi) - \tau')\,P(\xi)\,\mathrm{d}\xi \tag{25}$$

Introducing the autocorrelation function of the pulses with the same duration τ',

$$C(t,\tau') = \int \mathrm{d}\xi\,\delta(\tau(\xi) - \tau') \frac{P(\xi)}{P_\tau(\tau')} \int_0^{\tau'} \mathrm{d}t'\,A(t',\xi)\,A(t'+t,\xi), \tag{26}$$

we can write equation (24) in the form

$$C(t) = \nu \int P_\tau(\tau) \, C(t, \tau) \mathrm{d}\tau. \tag{27}$$

If the PDF of pulse durations has a power-law form

$$P_\tau(\tau) \sim \tau^\rho \tag{28}$$

and the autocorrelation function of the pulses with the same duration has the scaling property

$$C(at, a\tau) = a^{2\gamma+1} C(t, \tau) \tag{29}$$

then it follows that the autocorrelation function $C(t)$ has the required property (3) with β given by equation

$$\beta = \rho + 2\gamma + 3. \tag{30}$$

The meaning of the parameter γ will be revealed below, equations (35) and (36). Note, that the scaling in equation (29) is the same as described by equation (9) in [26], where the pulse area $S \sim \tau^{\gamma+1}$ is used instead of pulse duration τ.

In order to avoid the divergence of the PDF of pulse durations, equation (28) should hold only in some region of the pulse durations τ, $\tau_{\min} \leqslant \tau \leqslant \tau_{\max}$. In this case the scaling cannot be exact. However, if the influence of the limiting values $\tau_{\min}$ and $\tau_{\max}$ can be neglected for time t in some region $t_{\min} \ll t \ll t_{\max}$, we can expect that the scaling equation (3) approximately holds for this time region.

One of the ways to get the required scaling equation (29) of the autocorrelation function $C(t, \tau)$ is to consider pulses having the same shape, only stretched in height and in time. The signal consisting of such pulses was investigated in [21]. For stretched pulses we can write

$$A(t, \xi) = h(\xi) y(t / \tau(\xi)), \tag{31}$$

where $h(\xi)$ is the height of the pulse. The function $y(t_s)$ is nonzero only when $0 \leqslant t_s \leqslant 1$.

From equation (26) we obtain

$$C(t, \tau) = \tau \overline{h^2}(\tau) c(t / \tau), \tag{32}$$

where

$$\overline{h^2}(\tau') = \int \delta(\tau(\xi) - \tau') h^2(\xi) \frac{P(\xi)}{P_\tau(\tau')} \mathrm{d}\xi \tag{33}$$

is the mean squared amplitude of pulses having the same duration τ' and the function

$$c(t_s) = \int_0^1 y(t_s') y(t_s' + t_s) \mathrm{d}t_s' \tag{34}$$

is the autocorrelation function of pulse shapes. If the height h of the pulse is a power-law function of the pulse duration τ,

$$h \sim \tau^\gamma, \tag{35}$$

then the scaling equation (29) of the autocorrelation function (32) holds. It should be noted, that even when pulse height h is proportional to τ^γ, the coefficient of

Table 1. Some situations when the power-law dependence of the pulse height h on the pulse duration τ occurs. The corresponding power-law exponents γ together with the exponents ρ required to get $1/f$ PSD.

Signal	γ	ρ	Meaning of ρ
Constant area pulses	-1	0	Uniform distribution of pulse durations
Constant energy pulses	$-1/2$	-1	
Pulses of constant height	0	-2	Uniform distribution of inverse durations
Geometrically similar pulses	1	-4	

proportionality is not necessarily constant. In particular, the sign of the pulses can be random. Only coefficient of proportionality for the average of the square of the pulse height $\overline{h^2}(\tau) \sim \tau^{2\gamma}$ should be constant. If pulse height is a power-law function of the pulse duration, then the change of the magnitude of the pulse height $h \to ah$ is caused by the change of pulse duration $\tau \to a^{1/\gamma}\tau$. Comparing this scaling property to equation (6) we see that the power-law exponent γ plays a similar role as $-1/\mu$, i.e.,

$$\gamma = -\frac{1}{\mu} = \frac{1}{2(1-\eta)}. \tag{36}$$

The sign minus in equation (36) appears because stretching the time, as in equation (6), is equivalent to the shortening of the pulse duration.

Now we will investigate the influence of limiting pulse durations $\tau_{\min}$ and $\tau_{\max}$. From the assumptions made above, equations (28), (30) and (35), we have that $P_\tau(\tau)\,\overline{h^2}(\tau) = B\tau^{\beta-3}$ when $\tau_{\min} \leqslant \tau \leqslant \tau_{\max}$. Here B is the coefficient independent from τ. For time $\tau_{\min} \ll t \ll \tau_{\max}$ we can write the autocorrelation function, according to equations (27) and (32), as

$$C(t) = \nu B \int_{\tau_{\min}}^{\tau_{\max}} \tau^{\beta-2} c(t/\tau)\,\mathrm{d}\tau = \nu B t^{\beta-1} \int_{1}^{\frac{\tau_{\max}}{t}} u^{\beta-2} c(1/u)\,\mathrm{d}u. \tag{37}$$

According to paper [21], physically reasonable pulses are square-integrable and have everywhere a finite derivative of the autocovariance function with respect to time. Then for large u we can approximate $c(1/u) \approx c(0) + c'(0)/u$ and get

$$C(t) \approx \begin{cases} vBt^{\beta-1}\left[\dfrac{c(0)}{1-\beta} + \dfrac{c'(0)}{2-\beta}\right], & 0 < \beta < 1, \\[2mm] vB\big[c(0)\ln\tau_{\max} + c'(0) - c(0)\ln t\big], & \beta = 1, \\[2mm] vB\left[c(0)\dfrac{\tau_{\max}^{\beta-1}}{\beta-1} + t^{\beta-1}\dfrac{c'(0)}{2-\beta}\right], & 1 < \beta < 2. \end{cases} \tag{38}$$

Thus, for $0 < \beta < 2$ and $\tau_{\min} \ll t \ll \tau_{\max}$ the term containing time t has a scaling property the same as in equation (3), the limiting values of the pulse durations $\tau_{\min}$ and $\tau_{\max}$ do not influence the scaling of the autocorrelation function. On the other hand, if $\beta > 2$ then the influence of $\tau_{\max}$ becomes significant.

As has been pointed out in [45], the condition for $1/f$ spectrum, $\rho + 2\gamma + 2 = 0$, can be easily satisfied. The power-law dependence of the pulse height on the pulse duration can occur naturally. Various cases are listed in table 1. The value $\gamma = 0$ corresponds to the pulses of constant height; $\gamma = -1$ corresponds to constant area pulses. Geometrically similar pulses have $\gamma = 1$. If the energy is proportional to the square of the signal, the constant
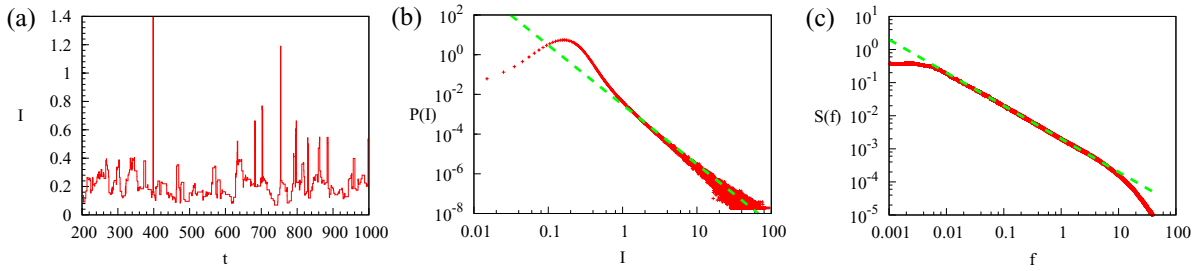
Figure 3. (a) Typical signal consisting of the constant area rectangular pulses with the uniformly distributed durations, $\rho = 0$. (b) The PDF of the signal intensity. The dashed (green) line shows the power-law with the exponent -3. (c) The PSD of such a signal. The dashed (green) line corresponds to the spectrum obtained using Wiener–Khintchine theorem from the autocorrelation function. The time intervals between the pulses are distributed according to Poisson process with the average $\overline{\vartheta} = 5$. The parameters used are $\tau_{\min} = 0.01$ and $\tau_{\max} = 100$.

energy pulses correspond to $\gamma = -1/2$. Since we have $1/f$ spectrum when $\rho = -2(\gamma+1)$, this spectrum occurs for constant area pulses ($\gamma = -1$) and uniform distribution of pulse durations ($\rho = 0$) in a wide interval. For constant height pulses ($\gamma = 0$) we have $1/f$ spectrum when the distribution of inverse durations τ^{-1} is uniform, that is when $P_\tau(\tau) \propto \tau^{-2}$.

A signal consisting of overlapping constant height pulses has a PDF of the Poisson distribution. On the other hand, pulses with $\gamma \neq 0$ can lead to power-law tails in the PDF of the signal. Let us consider rectangular pulses with the only random parameter being the pulse duration τ. Large signal intensities are due to pulses of large height, for which one can neglect the overlap between pulses. Each pulse of the height $h(\tau)$ occurs with the probability $P_\tau(\tau)$ and lasts for time τ. Thus, the PDF of the signal intensity $I = h$ is (see for analogy [34])

$$P_I(I) = \frac{\tau}{\langle \tau \rangle} P_\tau(\tau) \frac{\mathrm{d}\tau}{\mathrm{d}h}\bigg|_{h=I}. \tag{39}$$

If the PDF of pulse durations has the power-law $P_\tau(\tau) \propto \tau^\rho$ form and the height of the pulse depends on the pulse duration as $h \propto \tau^\gamma$, then from equation (39) we obtain $P_I(I) \propto I^{-\lambda}$, where

$$\lambda = 1 - \frac{2+\rho}{\gamma}. \tag{40}$$

For the pure $1/f$ noise $2+\rho = -2\gamma$ and we get the exponent $\lambda = 3$. For the case of $f^{-\beta}$ spectrum we have the following relation between the exponent β of the spectrum and exponent λ of the signal PDF:

$$\beta = 1 + \gamma(3 - \lambda). \tag{41}$$

Taking into account equation (36) we see that relation (41) is the same as equation (13), describing the power-law spectrum of the signal generated by the non-linear SDE (11). Note that the PDF of the signal intensity has the same power-law exponent $\lambda = 3$ also when $1/f$ noise is generated by the nonlinear SDE equation (11).

A typical signal for rectangular constant area pulses ($\gamma = -1$) is shown in figure 3(a), the PDF of the signal is shown in figure 3(b) and the PSD in figure 3(c). We see a good

agreement of numerically obtained PSD with the analytical estimation. In figure 3(b) we can see that the PDF has a power-law tail for large signal intensities. Note that, due to the overlapping of the pulses, the PDF of the signal at smaller intensities is not a power-law and acquires a power-law tail only for larger intensities, where the overlap can be neglected. This is in contrast with the SDE (11), where the steady state PDF of the signal can have a power-law form for all values of the signal sufficiently far from limiting values $x_{\min}$ and $x_{\max}$.

4. Connection between the nonlinear SDE and stochastic pulse sequences

As was shown in sections 2 and 3, the signals having $1/f^\beta$ PSD and generated by the nonlinear SDEs have similar scaling like the signals consisting from random pulses. In this section we approximate nonlinear SDE by a sequence of pulses and show that this approximation gives the same PDF of signal intensity and a power-law region in the PSD with the same exponent. However, other details of the approximated signal can be different: for example, a signal consisting of pulses does not exhibit the intermittent bursts characteristic for the solution of a SDE.

Let us consider SDE (11) together with reflective boundaries at $x = x_{\min}$ and $x = x_{\max}$. Using the Euler–Maruyama approximation with a time step $\Delta t = s$ and replacing the stochastic differential equation with the difference equation, we have

$$x_{k+1} = x_k + \sigma^2\left(\eta - \frac{\lambda}{2}\right)x_k^{2\eta-1}s + \sigma x_k^\eta \sqrt{s}\,\varepsilon_k. \tag{42}$$

Here ε_k is a Gaussian random variable with zero mean and unit variance. Variable time step [36]

$$s_k = \frac{\kappa^2}{\sigma^2}x_k^{2(1-\eta)} \tag{43}$$

results in the equation

$$x_{k+1} = x_k\left(1 + \frac{1}{2}\kappa^2(2\eta - \lambda) + \kappa\varepsilon_k\right), \tag{44}$$

where $\kappa \ll 1$ is a small parameter. The approximation of the SDE becomes better with decreasing κ. It should be noted that equation (44) has an universal form: x_k enters only in the first power independent on the exponent η. We can estimate the steady state PDF of x_k in k-space by considering equation (44) as an Euler–Maruyama approximation of the SDE in k-space,

$$\mathrm{d}x = \kappa^2\left(\eta - \frac{\lambda}{2}\right)x\,\mathrm{d}k + \kappa x\,\mathrm{d}W_k. \tag{45}$$

Taking into account reflective boundaries at $x_{\min}$ and $x_{\max}$ we get from the Fokker–Planck equation in k-space corresponding to equation (45) that the steady state PDF has the power-law form, $P(x_k) \propto x_k^{2(\eta-1)-\lambda}$. From this steady state PDF and equation (43) it follows that PDF of s_k has a power-law form $P_k(s_k) \propto s_k^\rho$ with

$$\rho = \frac{\lambda - 1}{2(\eta - 1)} - 2. \tag{46}$$

The same expression for the power-law exponent ρ can be obtained from equations (36) and (40).

From equation (44) we obtain x_{k+n}:

$$x_{k+n} = x_k \prod_{i=1}^{n} \left(1 + \frac{1}{2}\kappa^2(2\eta - \lambda) + \kappa\varepsilon_{k+i-1} \right) \approx x_k(1 + \kappa\sqrt{n}\,\varepsilon + \cdots), \tag{47}$$

where ε is a Gaussian random variable with zero mean and unit variance. Here we have used the fact that the sum of n Gaussian variables ε_{k+i-1} is a Gaussian variable with the dispersion equal to n. We can conclude that x_{k+n} does not differ significantly from x_k as long as $\kappa\sqrt{n} \ll 1$. The maximal value of n when x_{k+n} is approximately equal to x_k is $n_{\max} \sim 1/\kappa^2$. The duration in which the stochastic variable x does not change significantly is

$$\tau(x_k) = n_{\max}s_k = \frac{1}{\sigma^2}x_k^{2(1-\eta)} = \frac{1}{\sigma^2}x_k^{1/\gamma}, \tag{48}$$

where γ is given by equation (36). The duration τ, being proportional to s_k, has the power-law PDF with the same exponent ρ as the PDF of s_k:

$$P_\tau(\tau) = \begin{cases} C\tau^\rho, & \tau_{\min} \leq \tau \leq \tau_{\max}, \\ 0, & \text{otherwise.} \end{cases} \tag{49}$$

Here C is normalization coefficient and

$$\tau_{\min} = \frac{1}{\sigma^2 x_{\max}^{2(\eta-1)}}, \qquad \tau_{\max} = \frac{1}{\sigma^2 x_{\min}^{2(\eta-1)}}. \tag{50}$$

From equation (48) the value of the stochastic variable x_k is connected with the duration τ by the relation

$$x_k = \sigma^{2\gamma}\tau^\gamma. \tag{51}$$

Therefore, we can approximate the signal generated by SDE (11) by rectangular pulses of random duration τ having the PDF of durations equation (49) and pulse height $h \equiv x_k$ related to the pulse duration τ by equation (51). The pulses are not overlapping and immediately follow each other. Although the durations of adjacent pulses obtained from the signal generated by SDE equation (11) are correlated, for simplicity we will neglect this correlation. The PDF of the signal x constructed as such a pulse sequence has a power-law form. Using equations (40) and (46) we get that the power-law exponent in the PDF $P(x)$ is equal to $-\lambda$, with λ appearing in the SDE (11).

When pulses occur not randomly but follow each other, the other terms in the equation (23) are nonzero. However, one can check that for some range of time t the first part in the equation (23) dominates. Thus the PSD of this pulse sequence has a power-law part with the exponent given by equation (41). Using the value of the exponent γ from equation (36) we get the power-law exponent equation (13) in the PSD. The frequency range $\tau_{\max}^{-1} \ll f \ll \tau_{\min}^{-1}$ where the PSD of the signal, consisting of pulses,
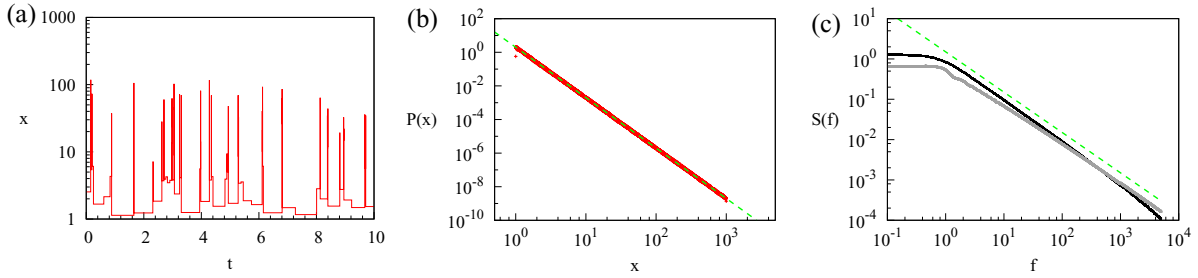
Figure 4. (*a*) Typical signal consisting of the equal energy pulses that immediately follow each other. The PDF of pulse durations τ is given by equation (49) with $\rho = -1$, $\tau_{\min} = 10^{-6}$ and $\tau_{\max} = 1$ and pulse heights are proportional to $\tau^{-1/2}$. (*b*) The PDF of the signal intensity. The dashed (green) line shows the power-law with the exponent -3. (*c*) Comparison of the PSD calculated using the signal generated by SDE (11) with the same parameters as in figure 1 (black line) and using the signal consisting of pulses (gray line). The dashed (green) line shows the slope $1/f$.

has power-law behavior coincides with inequalities equation (19). Thus, the proposed approximation of the SDE by a sequence of pulses correctly reproduces power-law parts of the PDF and the PSD of the generated signal.

To illustrate the approximation of nonlinear SDE by a sequence of pulses, in figure 4 we compare the PDF and the PSD of the signal consisting from pulses to the PDF and the PSD of the signal shown in figure 1. The SDE has parameters $\eta = 2$, $\lambda = 3$, $x_{\min} = 1$, $x_{\max} = 1000$, therefore, duration of the pulses has power-law PDF equation (49) with $\rho = -1$ (according to equation (46)) and $\tau_{\min} = 10^{-6}$, $\tau_{\max} = 1$. The height of each pulse is proportional to the duration of the pulse to the power of $\gamma = -1/2$, obtained from equation (36). This value of γ means that each pulse has the same energy. The signal consisting of such pulses is shown in figure 4(*a*). As one can see, this signal looks rather different from the one shown in figure 1(*a*). Large values of the signal in figure 4(*a*) do not come in intermittent bursts as in figure 1(*a*). This difference is caused by the assumption that the durations of different pulses are uncorrelated. The PDF of the signal, shown in figure 4(*b*), is the same as in figure 1(*b*). Comparison of the PSDs is shown in figure 4(*c*). There is qualitative agreement between the PSD of the signal generated by the nonlinear SDE and the PSD of the signal consisting of pulses. The PSD of the signal consisting of pulses has a power-law part in a different range of frequencies, from $f \approx 10^1$ up to $f \approx 4 \times 10^4$. This difference from the expected range $\tau_{\max}^{-1} < f < \tau_{\min}^{-1}$ is caused by other terms neglected in equation (23).

5. Conclusions

In summary, we have demonstrated the connection between nonlinear SDEs generating signals with $1/f^\beta$ noise and signals consisting of random pulses with the power-law distribution of pulse durations. The exponent ρ of the power-law PDF of pulse durations and the exponent γ characterizing the dependence of the pulse height on the pulse duration are related to the parameters η and λ of the SDE (11) by means of the equations (36) and (46). The signal generated by the SDE and the corresponding signal consisting of

rectangular pulses, yield the same distribution of signal intensity and the same power-law exponent in the PSD equation (13). The appearance of $1/f^\beta$ spectrum and relationship between parameters can be obtained just by considering the scaling properties of the signals. The revealed connection between different models of $1/f$ noise provides further insights into the origin and relationship between different models of $1/f$ noise.

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