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On the intrinsic origin of $1/f$ noise

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Abstract

The problem of the intrinsic origin of $1/f$ noise is considered. Currents and signals consisting of a sequence of pulses are analyzed. It is shown that the intrinsic origin of $1/f$ noise is a random walk of the average time between subsequent pulses of the pulse sequence, or the interevent time. This results in the long-memory process for the pulse occurrence time and in $1/f$ type power spectrum of the signal. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

Ubiquitous signals and processes with $1/f$ power spectral density at low frequencies has led to speculations that there might exist some generic mechanism under the production of $1/f$ noise. The generic origins of two popular noises, white noise (no correlation in time, $S(f) \propto 1/f^0$) and Brownian noise (no correlation between increments, $S(f) \propto 1/f^2$) are very well known. It should be noted that Brownian motion is the integral of white noise and that the operation of integration of the signal increases the exponent by 2 while the inverse operation of differentiation decreases it by 2. Therefore, $1/f$ noise cannot be obtained by the simple procedure of integration or differentiation of the convenient signals. There are no simple, even stochastic, equations generating signals with $1/f$ noise. Also, note the concept of the fractional Brownian motion and the half-integration of white noise signals used for the generation of processes with $1/f$ noise [1] in this context. These and similar mathematical algorithms, procedures and models for the generation of the processes with $1/f$ noise [2,3] are, however, sufficiently specific, formal or unphysical. They cannot, as a rule, be solved analytically and they do not reveal the origin as well as the necessary and sufficient conditions for the appearance of $1/f$ type fluctuations. Physical models of $1/f$ noise in some physical systems are usually very specialized and complicated and they do not explain the internal origin of the omnipresent processes with $1/f^\delta$ spectrum.

A lot of contributions are available in the literature concerning the origin of $1/f$ noise. On the web (), Wentain Li is the collective bibliography of flicker noise. Sufficiently comprehensive bibliography of the contributions concerning the modeling of $1/f$ noise may be found in Refs. [3–10].

This work is a continuation of series of papers devoted to modeling $1/f$ noise in simple systems [4–6] and searches the necessary conditions for the appearance of the signals with power spectrum at low frequencies like $S(f) \propto 1/f^\delta$ ($\delta \simeq 1$) [7–12]. In the Refs. [4–6], an analysis of the necessary conditions for the appearance of $1/f$ type fluctuations in the simple systems consisting of few or even one particle and affected by random perturbations is presented. Later, a simple, analytically solvable model of $1/f$ noise has been proposed [7,8], analyzed [9,10] and generalized [11]. The model reveals the main features and the parameter dependencies of the power spectrum of $1/f$ noise.

Here, considering signals and currents as consisting of pulses, generalizations and development of the model [7,8] are presented. The paper includes derivation of the expression for the correlation function, analysis of the examples of different signals and exhibition of the necessary conditions for the appearance of $1/f$ type power spectrum in the signals consisting of pulses. It is shown that the intrinsic origin of $1/f$ noise is a Brownian motion of the pulse interevent time.

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2. The model

Let us consider currents or signals represented as sequences of random (but correlated) pulses $A_k(t-t_k)$. Function $A_k(t-t_k)$ represents the shape of the k-pulse of the signal in the region of the pulse occurrence time t_k. The signal or intensity of the current of particles in some space cross-section may, therefore, be expressed as

$$I(t)=\sum_k A_k(t-t_k). \tag{1}$$

It is easy to show that the shapes of the pulses mainly influence the high frequency, $f \geqslant \Delta t_p$ with Δt_p being the characteristic pulse length, power spectrum while fluctuations in the pulse amplitudes result, as a rule, in the white or Lorentzian but not $1/f$ noise [13]. Therefore, we restrict our analysis to the noise due to the correlations between the pulse occurrence times t_k. In such an approach, we can replace the function $A_k(t-t_k)$ by $a\delta(t-t_k)$. Here $\delta(t-t_k)$ is the Dirac delta function, $a=\langle\int_{-\infty}^{+\infty}A_k(t-t_k)\,\mathrm{d}t\rangle$ is the average area of the pulse and the brackets $\langle\cdots\rangle$ denote the averaging over realizations of the process. In such an approach, signal (1) may be expressed as

$$I(t)=a\sum_k\delta(t-t_k). \tag{2}$$

This model also corresponds to the flow of identical point objects such as electrons, photons, cars and so on. On the other hand, fluctuations in the amplitudes A_k may result in the additional noise but cannot reduce $1/f$ noise we are looking for.

2.1. Power spectrum

Power spectral density of the signal $I(t)$ is

$$S(f)=\lim_{T\to\infty}\left\langle\frac{2}{T}\left|\int_{t_i}^{t_f}I(t)\mathrm{e}^{-\mathrm{i}2\pi ft}\mathrm{d}t\right|^2\right\rangle, \tag{3}$$

where $T=t_f-t_i$ is the observation time.

We can also introduce the autocorrelation function,

$$\Phi(s)=\left\langle\frac{1}{T}\int_{t_i}^{t_f-s}I(t)I(t+s)\,\mathrm{d}t\right\rangle \tag{4}$$

and use the Wiener–Khintchine relations,

$$S(f)=4\lim_{T\to\infty}\int_0^T\Phi(s)\cos(2\pi fs)\,\mathrm{d}s,\qquad \Phi(s)=\int_0^\infty S(f)\cos(2\pi fs)\,\mathrm{d}f. \tag{5}$$

Substitution of Eq. (2) into Eq. (3) results in the power spectral density of the signal expressed as sequence of pulses,

$$S(f)=\lim_{T\to\infty}\left\langle\frac{2a^2}{T}\left|\sum_{k=k_{\min}}^{k_{\max}}\mathrm{e}^{-\mathrm{i}2\pi ft_k}\right|^2\right\rangle =\lim_{T\to\infty}\left\langle\frac{2a^2}{T}\sum_{k=k_{\min}}^{k_{\max}}\sum_{q=k_{\min}-k}^{k_{\max}-k}\mathrm{e}^{\mathrm{i}2\pi f\Delta(k;q)}\right\rangle, \tag{6}$$

where $\Delta(k;q)\equiv t_{k+q}-t_k$ is the difference of pulse occurrence times t_{k+q} and t_k while $k_{\min}$ and $k_{\max}$ are minimal and maximal values of index k in the interval of observation $T=t_f-t_i$.

For a stationary process, Eq. (6) yields

$$S(f)=\frac{2a^2}{T}\sum_{q=-N}^{N}(N+1-|q|)\left\langle\mathrm{e}^{\mathrm{i}2\pi f\Delta(q)}\right\rangle.$$

Here $N=k_{\max}-k_{\min}$, the brackets $\langle\cdots\rangle$ denote the averaging over realizations of the process and over the time (index k) and a definition $\Delta(q)=-\Delta(-q)=\Delta(k;q)$ is introduced. To abbreviate the equations, we have omitted the mark of the limit "$\lim_{T\to\infty}$" here and further, in expressions for the power spectrum $S(f)$.

The average over the distribution of $\Delta(q)$ may be expressed as

$$\left\langle\mathrm{e}^{\mathrm{i}2\pi f\Delta(q)}\right\rangle\equiv\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}2\pi f\Delta(q)}\Psi_\Delta(\Delta(q))\mathrm{d}\Delta(q)=\chi_{\Delta(q)}(2\pi f).$$

Here $\Psi_\Delta(\Delta(q))$ is the distribution density of $\Delta(q)$ and $\chi_{\Delta(q)}(2\pi f)$ is the characteristic function of the distribution $\Psi_\Delta(\Delta(q))$. Therefore,

$$S(f)=2a^2\sum_{q=-N}^{N}\left(\bar{\nu}-\frac{|q|}{T}\right)\chi_{\Delta(q)}(2\pi f), \tag{7}$$

where $\bar{\nu}=\langle\lim_{T\to\infty}(N+1)/T\rangle$ is the mean number of pulses per unit time.

When the sum $\sum_{q=-N}^{N}|q|\chi_{\Delta(q)}(2\pi f)$ converges and $T\to\infty$, we have the power spectrum from Eq. (7) in the form

$$S(f)=2a\bar{I}\sum_{q=-N}^{N}\chi_{\Delta(q)}(2\pi f). \tag{8}$$

Here $\bar{I}\equiv\langle\overline{I(t)}\rangle=\bar{\nu}a$ is the average current.

2.2. Correlation function

Substitution of Eq. (2) into Eq. (4) yields the correlation function of signal (2)

$$\Phi(s)=\left\langle\frac{a^2}{T}\sum_{k,q}\delta\left(t_{k+q}-t_k-s\right)\right\rangle.$$

After summation over index k, we have

$$\Phi(s)=\bar{I}a\sum_q\langle\delta(\Delta(q)-s)\rangle,$$

where the brackets $\langle\cdots\rangle$ denote again the averaging over realizations of the process and over the time (index k) as well. Such averaging coincides with the averaging over the distribution of the time difference $\Delta(q)$.

$$\Phi(s) = \bar{I}a\sum_q \int_{-\infty}^{+\infty} \psi_\Delta(\Delta)\delta(\Delta - s)\,\mathrm{d}\Delta,$$
$$\Phi(s) = \bar{I}a\delta(s) + \bar{I}a\sum_{q\neq 0}\psi_\Delta(s). \tag{9}$$

Here $\psi_\Delta(\Delta)$ is the distribution density of $\Delta(q)$. Substitution of Eq. (9) into Eq. (5) yields expressions (7) and (8).

3. Examples

Consider some examples of the signals represented by Eq. (2).

(i) Periodic signal expressed as $I(t) = a\sum_k \delta(t - k\tau)$ generates the power spectrum

$$S(f) = 2a^2 \lim_{T\to\infty} \frac{\sin^2(\pi(N+1)\tau f)}{T\sin^2(\pi\tau f)} \Rightarrow 2\bar{I}^2\delta(f), \quad f \ll \tau^{-1}.$$

(ii) Perturbed periodic signal represented by Eq. (2) with the times series expressed as recurrence equations $t_k - t_{k-1} \equiv \tau_k = \bar{\tau} + \sigma\varepsilon_k$ with $\{\varepsilon_k\}$ being a sequence of uncorrelated normally distributed random variables with zero expectation and unit variance and σ being the standard deviation of this white noise [5,6]. For this model, we have $\Delta(q) = q\bar{\tau} + \sigma\sum_{l=k+1}^{k+q}\varepsilon_l$ and the characteristic function,

$$\chi_{\Delta(q)}(2\pi f) = \exp\left(\mathrm{i}2\pi f\langle\Delta(q)\rangle - \frac{1}{2}(2\pi f)^2\sigma_\Delta^2\right), \tag{10}$$

where $\langle\Delta(q)\rangle = q\bar{\tau}$ and the variance σ_Δ^2 of the time difference $\Delta(q)$ equals $\sigma_\Delta^2 \equiv \left\langle \Delta(q)^2\right\rangle - \langle\Delta(q)\rangle^2 = \sigma^2|q|$. Substitution of Eq. (10) into Eq. (8) yields the Lorentzian spectrum,

$$S(f) = \bar{I}^2\frac{4\tau_{\mathrm{rel}}}{1 + \tau_{\mathrm{rel}}^2\omega^2}, \tag{11}$$

where $\omega = 2\pi f$ and $\tau_{\mathrm{rel}} = \sigma^2/2\bar{\tau}$.

(iii) Time difference $\Delta(q) = \sum_{l=k+1}^{k+q}\tau_l$ as a sum of uncorrelated interevent times τ_l. According to Eqs. (6)–(8), we have in this case

$$S(f) = 2a\bar{I}\left[1 + 2\mathrm{Re}\sum_{q=1}^{N}\left\langle \mathrm{e}^{\mathrm{i}2\pi f\tau}\right\rangle^q\right],$$
$$S(f) = 2a\bar{I}\left[1 + 2\mathrm{Re}\frac{\chi_\tau(\omega)}{1 - \chi_\tau(\omega)}\right]. \tag{12}$$

For instance, substitution at $f \ll \bar{\tau}^{-1}$ and $f \ll \sigma^{-1}$ of Eq. (10) with $q = 1$ into Eq. (12) results in Eq. (11).

(iv) For the Poisson process,

$$\chi_\tau(2\pi f) = \frac{1}{1 - \mathrm{i}2\pi f\bar{\tau}}, \qquad \mathrm{Re}\frac{\chi_\tau(\omega)}{1 - \chi_\tau(\omega)} = 0,$$

and we have from Eq. (12) only the shot noise,

$$S(f) = 2a\bar{I} = S_{\mathrm{shot}}. \tag{13}$$

(v) Brownian motion of the interevent time τ_k with some restrictions, e.g., with the relaxation to the average value $\bar{\tau}$,

$$\tau_k = \tau_{k-1} - \gamma(\tau_{k-1} - \bar{\tau}) + \sigma\varepsilon_k, \tag{14}$$

when the pulse occurrence times t_k are expressed as

$$t_k = t_{k-1} + \tau_k. \tag{15}$$

According to Eq. (6), the power spectrum of signal (2) with the pulse occurrence times t_k generated by Eqs. (14) and (15) for sufficiently small parameters σ and γ in any desirably wide range of frequencies, $f_1 = \gamma/\pi\sigma_\tau < f < f_2 = 1/\pi\sigma_\tau$, is $1/f$-*like* [7–12], i.e.,

$$S(f) = \bar{I}^2\sqrt{\frac{2}{\pi}}\frac{\bar{\tau}\exp\left(-\bar{\tau}^2/2\sigma_\tau^2\right)}{\sigma_\tau f}. \tag{16}$$

Here $\sigma_\tau^2 = \sigma^2/2\gamma$ is the variance of the interevent time τ_k.

4. Origin of $1/f$ noise

The origin for appearance of $1/f$ fluctuations in the model described in Eqs. (14) and (15) is related with the relatively slow, Brownian, fluctuations of the pulse interevent time. For this reason, the variance σ_Δ^2 of the time difference $\Delta(k;q)$ for $|q| \ll \gamma^{-1}$ is a quadratic function of the time difference and, consequently, of the difference q of the pulse serial numbers k [7–12], i.e.,

$$\sigma_\Delta^2(k;q) = \sigma_\tau^2(k)q^2. \tag{17}$$

Substitution of Eqs. (10) and (17) into Eq. (8) yields $1/f$ spectrum (16).

4.1. Generalization

For slowly fluctuating interevent time, the time difference $\Delta(k;q)$ may be expressed as [7–12]

$$\Delta(q) = \sum_{l=k+1}^{k+q}\tau_l \simeq q\tau, \tag{18}$$

where $\tau = \left(t_{k+q} - t_k\right)/q$ is the average interevent time in the time interval $\left(t_k, t_{k+q}\right)$, a slowly fluctuating function of the arguments k and q. In such an approach, the power spectrum according to Eq. (6) is

$$S(f) = 2\bar{I}a\sum_q\left\langle \mathrm{e}^{\mathrm{i}2\pi fq\tau}\right\rangle, \tag{19}$$

where

$$\langle e^{i2\pi fq\tau} \rangle \equiv \int_{-\infty}^{+\infty} e^{i2\pi fq\tau} \psi_\tau(\tau) d\tau = \chi_\tau(2\pi fq)$$

is the characteristic function of the distribution density $\psi_\tau(\tau)$ of the interevent time τ. Therefore, the power spectrum according to Eq. (19) may be expressed as

$$S(f) \simeq 2\bar{I}^2 \bar{\tau} \Psi_\tau(0)/f. \quad (20)$$

Here, the property $\int_{-\infty}^{+\infty} \chi_\tau(x) dx = 2\pi \Psi_\tau(0)$ of the characteristic function has been used.

4.2. Correlation function of $1/f$ noise

The correlation function of $1/f$ noise in approximation (18) may be calculated according to Eq. (9), i.e.,

$$\Phi(s) = \bar{I}a \sum_q \int_{-\infty}^{+\infty} \psi_\tau(\tau) \delta(q\tau - s) d\tau$$
$$= \bar{I}a\delta(s) + \bar{I}a \sum_{q \neq 0} \psi_\tau\left(\frac{s}{q}\right) \frac{1}{|q|}. \quad (21)$$

For the Gaussian distribution of the interevent time τ,

$$\psi_\tau(\tau) = \frac{1}{\sqrt{2\pi}\sigma_\tau} \exp\left(-\frac{(\tau - \bar{\tau})^2}{2\sigma_\tau^2}\right),$$

the correlation function (21) reads as

$$\Phi(s) = \frac{\bar{I}a}{\sqrt{2\pi}\sigma_\tau} \sum_q e^{-(s-q\bar{\tau})^2/2\sigma_\tau^2 q^2} \frac{1}{|q|}. \quad (22)$$

It should be noted that the deviation of the variance σ_Δ^2 for large q from the quadratic dependence (17) and the approach to the linear function $\sigma_\Delta^2 = 2D_{t_k}|q|$ ensures the convergence of sums (21) and (22) and, consequently, results in the Lorentzian power spectrum (11) at $f \to 0$ [7–12]. Here D_{t_k} is the "diffusion" coefficient of the pulse occurrence time t_k, related with the variance $\sigma_{t_k}^2$ of the pulse occurrence time as $\sigma_{t_k}^2 = 2D_{t_k}k$. For models (14)–(15), $D_{t_k} = \sigma^2/2\gamma^2$. The power spectra calculated according to Eq. (5) with the correlation functions (21) and (22) are expressed as Eqs. (20) and (16), respectively.

5. Conclusions

From the above analysis, we can conclude that the intrinsic origin of $1/f$ noise is the Brownian fluctuations of the interevent time of the signal pulses, similar to the Brownian fluctuations of the signal amplitude resulting in $1/f^2$ noise. The random walk of the interevent time in the time axis is a property of the randomly perturbed or complex systems with the elements of self-organization.

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