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Landau levels of cold atoms in non-Abelian gauge fields

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Abstract. The Landau levels of cold atomic gases in non-Abelian gauge fields are analyzed. In particular we identify effects on the energy spectrum and density distribution which are purely due to the non-Abelian character of the fields. We investigate in detail non-Abelian generalizations of both the Landau and the symmetric gauge. Finally, we discuss how these non-Abelian Landau and symmetric gauges may be generated by means of realistically feasible lasers in a tripod scheme.

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Contents

1. Introduction

Gauge potentials are crucial for the understanding of fundamental forces between subatomic particles. A simple example of a gauge potential is provided by the vector potential in the theory of electromagnetism [1]. In this example the different vector components are scalars, and hence they commute with each other. If the vector components of the gauge field are not scalars, but instead $N \times N$ matrices, with $N > 1$, then it is in principle possible to have a situation where the different vector components do not commute. However, non-Abelian gauge fields are scarce in nature. Candidates so far have mainly been restricted to molecular systems [2] which are largely approachable only through spectroscopic means. Other systems are liquid crystals which show the required non-Abelian symmetries [3].

Experiments on cold quantum gases have reached an unprecedented degree of control, offering thus extraordinary possibilities for the analysis of the effects of gauge fields on atomic systems. A simple way of generating a gauge field in ultracold gases is by rotating Bose–Einstein condensates with an angular frequency $\Omega\hat{z}$ (where we employ cylindrical coordinates $\{\rho, \varphi, z\}$). In the corresponding rotating frame the Hamiltonian describing the rotating system becomes one of a system subject to a symmetric gauge field $\vec{A} = -m\Omega\rho\hat{\varphi}$, where m is the atomic mass [4]. Thus, a rotating condensate resembles a gas under the influence of a constant magnetic field $B_0 = m\Omega$, and as a consequence many interesting phenomena, including, e.g., Landau level physics and quantum-Hall-like phenomena have been studied in rotating quantum gases [5]–[8].

Due to their internal structure, ultracold atoms offer as well the possibility of creating non-Abelian gauge fields. A surprising and astoundingly elegant derivation and description of the emergence of non-Abelian gauge potentials was presented by Wilczek and Zee [9]. It was shown by these authors that in the presence of a general adiabatic motion of a quantum system with degenerate states, gauge potentials will appear which are traditionally only encountered in high energy physics to describe the interactions between elementary particles. Ultracold atomic clouds are particularly promising candidates for realizing such scenarios, since the access to physical parameters is, from an experimental point of view, unprecedented. Extending the ideas of Wilczek and Zee, it was recently proposed that properly tailored laser beams coupled to degenerate internal electronic states can be employed to induce Abelian as well as non-Abelian gauge fields in cold-atom experiments [10]–[13]. Alternatively, such gauge potentials can be constructed in an optical lattice using laser assisted state sensitive tunneling [14]–[17].

With the implementation of these proposals, ultracold atoms would offer a unique testbed for the analysis of nontrivial effects on the properties of multicomponent cold atomic systems in the presence of non-Abelian gauge fields. To the best of our knowledge these effects have been scarcely studied in the literature [12, 13, 15, 18]. This paper is devoted to the analysis of non-Abelian effects on the spectral properties of ultracold atomic systems. In particular, we show how purely non-Abelian effects lead to the eventual destruction of the Landau level structure, and may significantly modify the ground state density profile of ideal quantum gases.

The structure of the paper is as follows: in section 2 we study the generation of different forms of non-Abelian gauge fields, including non-Abelian constant fields, as well as the non-Abelian generalization of the Landau gauge. Section 3 is devoted to the analysis of constant non-Abelian gauge fields. Section 4 discusses the non-Abelian Landau gauge, and in particular the destruction of the Landau level structure and the corresponding modified de Haas–van Alphen effect. In Section 5, we discuss the non-Abelian symmetric gauge. Finally in section 6, we conclude and discuss some future promising directions.

2. Laser-induced non-Abelian gauge fields

In this section, we discuss the generation of (possibly non-Abelian) gauge fields as those discussed in the following sections. There are at least two alternatives for the creation of non-Abelian gauge fields. One consists of employing two-component atoms in state-dependent optical lattices in the presence of appropriate laser arrangements [14, 15]. A second possibility, which we shall explore in this paper, was recently proposed in [12].

In this second alternative, a non-Abelian gauge potential is constructed for atoms with a tripod electronic structure [19, 20][5]. Three lasers with appropriate polarizations couple the excited electronic state $|0\rangle$ and the ground states $|j=1,2,3\rangle$, with corresponding Rabi frequencies $\Omega_j(\vec{r})$ which parametrize in the form: $\Omega_1=\Omega\sin\theta\cos\phi\, e^{iS_1}$, $\Omega_2=\Omega\sin\theta\sin\phi\, e^{iS_2}$, $\Omega_3=\Omega\cos\theta\, e^{iS_3}$. For a fixed position $\vec{r}$ the Hamiltonian describing the laser–atom interaction may be diagonalized to give a set of dressed states. Under appropriate adiabatic conditions two dressed states, so-called dark states, become decoupled from the other states:

$$|D_1\rangle=\sin\phi e^{iS_{31}}|1\rangle-\cos\phi e^{iS_{32}}|2\rangle, \tag{1}$$

$$|D_2\rangle=\cos\theta\cos\phi e^{iS_{31}}|1\rangle+\cos\theta\sin\phi e^{iS_{32}}|2\rangle-\sin\theta|3\rangle, \tag{2}$$

with $S_{ij}=S_i-S_j$. The dark states have zero eigenvalues and are separated by the energy $\hbar\Omega$ from the remaining eigenstates. The adiabatic approximation is justified if Ω is sufficiently large compared to the two-photon detuning due to the laser mismatch and/or Doppler shift. In that case the internal state of an atom evolves within the dark state manifold. The state of the atom

[5] For example, using the transition $2^3S_1\leftrightarrow 2^3P_0$ in $^4He^*$, or transition $5S_{1/2}$ $(F=1)\leftrightarrow 5P_{3/2}$ $(F=0)$ in ^{87}Rb.

can therefore be expanded in the dark state basis as $|\Phi\rangle = \Psi_1(\vec{r})\,|D_1(\vec{r})\rangle + \Psi_2(\vec{r})\,|D_2(\vec{r})\rangle$. The two-component spinor $\vec{\Psi} = \{\Psi_1, \Psi_2\}^T$ obeys a spinor Schrödinger equation of the form

$$i\hbar\frac{\partial}{\partial t}\vec{\Psi} = \left[\frac{1}{2m}(-i\hbar\vec{\nabla} - \hat{A})^2 + \hat{V} + \hat{\Phi}\right]\vec{\Psi}. \tag{3}$$

In this equation we observe the appearance of a 2×2 vector potential of the form

$$\begin{aligned}
\hat{A}_{11} &= \hbar\left(\cos^2\phi\vec{\nabla}S_{23} + \sin^2\phi\vec{\nabla}S_{13}\right),\\
\hat{A}_{12} &= \hbar\cos\theta\left(\tfrac{1}{2}\sin(2\phi)\vec{\nabla}S_{12} - i\vec{\nabla}\phi\right),\\
\hat{A}_{22} &= \hbar\cos^2\theta\left(\cos^2\phi\vec{\nabla}S_{13} + \sin^2\phi\vec{\nabla}S_{23}\right).
\end{aligned} \tag{4}$$

The systems also present a scalar potential of the form $\Phi_{ij} = \frac{\hbar^2}{2m}\vec{\kappa}_i^* \cdot \vec{\kappa}_j$, where

$$\vec{\kappa}_1 = \sin\theta\left(\tfrac{1}{2}\sin(2\phi)\vec{\nabla}S_{12} + i\vec{\nabla}\phi\right), \tag{5}$$

$$\vec{\kappa}_2 = \tfrac{1}{2}\sin(2\theta)(\cos^2\phi\vec{\nabla}S_{13} + \sin^2\phi\vec{\nabla}S_{23}) - i\vec{\nabla}\theta. \tag{6}$$

Finally, if the original states $|j = 1, 2, 3\rangle$ experience an external potential $U_j(\vec{r})$, then

$$V_{11} = U_2\cos^2\phi + U_1\sin^2\phi, \tag{7}$$

$$V_{12} = \frac{U_1 - U_2}{2}\cos\theta\sin(2\phi), \tag{8}$$

$$V_{22} = (U_1\cos^2\phi + U_2\sin^2\phi)\cos^2\theta + U_3\sin^2\theta. \tag{9}$$

Recent advances in shaping both the phase and the intensity of light beams make it possible to achieve a remarkable versatility in controlling the gauge fields [21, 22], provided the corresponding light fields obey Maxwell's equations.

In the following we shall assume that the atoms are strongly trapped in the z-direction, hence they are confined to the xy-plane. Given two orthogonal vectors $\vec{\xi}$ and $\vec{\eta}$ on the xy-plane, we shall be interested in non-Abelian situations, in which $\hat{A}_\xi \equiv \hat{A}\cdot\vec{\xi}$ and $\hat{A}_\eta \equiv \hat{A}\cdot\vec{\eta}$, fulfill $[\hat{A}_\xi, \hat{A}_\eta] \neq 0$. This condition demands $(\vec{u}\times\vec{\nabla}\phi)_z \neq 0$ and/or $(\vec{u}\times\vec{\nabla}S_{12})_z \neq 0$, and/or $(\vec{\nabla}S_{12}\times\vec{\nabla}\phi)_z \neq 0$, with $\vec{u} = (\cos^2\phi - \cos^2\theta\sin^2\theta)\vec{\nabla}S_{23} + (\sin^2\phi - \cos^2\theta\cos^2\phi)\vec{\nabla}S_{13}$.

2.1. Constant intensities

We will consider first homogeneous intensity profiles, i.e. both ϕ and θ are now space independent. We choose the particular case with $\phi = \theta = \pi/4$. For constant ϕ the non-Abelian character demands $\vec{\nabla}S_{23}\times\vec{\nabla}S_{13} \neq 0$. A simple laser arrangement fulfilling this condition is $S_{j3} = \alpha_j x + \beta_j y$, where α_j, β_j are constants such that $\alpha_2\beta_1 \neq \alpha_1\beta_2$. The corresponding x and y components of the vector potential are of the form

$$\hat{A}_x = \frac{1}{8}(\alpha_1 + \alpha_2)(3\hat{1} + \hat{\sigma}_z) + \frac{1}{2\sqrt{2}}(\alpha_1 - \alpha_2)\hat{\sigma}_x, \tag{10}$$

$$\hat{A}_y = \frac{1}{8}(\beta_1 + \beta_2)(3\hat{1} + \hat{\sigma}_z) + \frac{1}{2\sqrt{2}}(\beta_1 - \beta_2)\hat{\sigma}_x. \tag{11}$$

On the other hand, by choosing $V_j(\vec{r}) = \Delta E_j + U(\vec{r})$, with $\Delta E_1 = -(\hbar^2/16m)[(\alpha_1^2 - \alpha_2^2) + (\beta_1^2 - \beta_2^2)] = -\Delta E_2$, and $\Delta E_3 = -(\hbar^2/16m)[(\alpha_1^2 + \alpha_2^2) + (\beta_1^2 + \beta_2^2)]$, one can prove that (up to an irrelevant constant) $\hat{V} + \hat{\phi} = U(\vec{r})$ with $U(\vec{r})$ a common trapping potential for all components.

A gauge transformation eliminates the terms proportional to the identity matrix in $\hat{A}_x$ and $\hat{A}_y$. Let $\hbar\kappa_y = (\beta_1 - \beta_2)/2\sqrt{2}$, $\hbar q_y = (\beta_1 + \beta_2)/8$, $\hbar\kappa_x = (\alpha_1 - \alpha_2)/2\sqrt{2}$ and $\hbar q_y = (\alpha_1 + \alpha_2)/8$. A rotation $\hat{\sigma}_x \to \cos\eta\hat{\sigma}_x + \sin\eta\hat{\sigma}_z$, $\hat{\sigma}_x \to -\sin\eta\hat{\sigma}_x + \cos\eta\hat{\sigma}_z$, with $\tan 2\eta = \kappa_y/q_y$, provides $\hat{A}_y = \hbar\tilde{q}_y\hat{\sigma}_z$, with $\tilde{q}_y = \cos 2\phi q_y + \sin 2\phi\kappa_y$, and $\hat{A}_x = \hbar\tilde{\kappa}_x\hat{\sigma}_x + \hbar\tilde{q}_x\hat{\sigma}_z$, with $\tilde{\kappa}_x = (\cos 2\phi\kappa_x - \sin 2\phi q_x)$ and $\tilde{q}_x = (\cos 2\phi q_x + \sin 2\phi\kappa_x)$. Hence, we recover exactly the same form as discussed in section 3.

2.2. Landau-like gauge

In this subsection we shall consider the case $S_{13} = S_{23} = S$. In that case the non-Abelian character demands $(\vec{\nabla}S \times \vec{\nabla}\phi)_z \neq 0$. We will choose the phase $S = \kappa x$, and $\phi = qy$, which gives a non-Abelian gauge potential unless $\kappa = 0$ or $q = 0$. In addition we take $\cos\theta = x/R_c$, where $R_c^2 = x^2 + (z - z_c)^2$, such that for the relevant x-range, $|x| \ll z_c$ is fulfilled. As a consequence, and up to first order in (x/z_c) we obtain:

$$\hat{A} \simeq \hbar\kappa(\hat{1} + \hat{\sigma}_z)\hat{x} + B_0 x\hat{\sigma}_y\hat{y}. \tag{12}$$

where $B_0 = q/z_c$. Note that although $x \ll z_c$, B_0 can actually have large values. In addition, and again up to first order in (x/z_c), we obtain $\hat{V} + \hat{\phi} = U(\vec{r})$, if $V_1(\vec{r}) = V_2(\vec{r}) = \hbar^2 q^2/2m + U(\vec{r})$ and $V_3(\vec{r}) = \hbar^2/2mz_c^2$. Using a simple gauge transformation $\Psi \to \exp i\kappa x\Psi$ to eliminate the identity matrix term in $\hat{A}_x$, and applying a unitary spin transformation $U^\dagger\hat{A}U$, with $U = (\hat{\sigma}_z + \hat{\sigma}_y)/\sqrt{2}$, we obtain $\hat{A} \simeq \hbar\kappa\hat{\sigma}_y\hat{x} + B_0 x\hat{\sigma}_z\hat{y}$, which is indeed exactly the same Landau-like gauge that we employ in section 4. A simple laser arrangement which would lead to this particular gauge is provided by

$$\Omega_1 = \Omega\cos qy \mathrm{e}^{\mathrm{i}\kappa(x+y+z)/2}, \tag{13}$$

$$\Omega_2 = \Omega\sin qy \mathrm{e}^{\mathrm{i}\kappa(x+y+z)/2}, \tag{14}$$

$$\Omega_3 = \Omega\frac{x}{z_c}\mathrm{e}^{\mathrm{i}\kappa(x-y+z)/2}, \tag{15}$$

where we assume the illuminated atoms are confined to a region for which $|x| \ll z_c$ holds.

3. Constant non-Abelian gauge

Let us consider a constant matrix gauge of the form $\hat{A} = (\hat{A}_x, \hat{A}_y, 0)$. We have already shown that these fields can be generated in a tripod scheme using a simple laser arrangement. Then, the Hamiltonian of the 2D system becomes:

$$\hat{H} = \frac{1}{2m}\left[(\hat{p}_x + \hat{A}_x)^2 + (\hat{p}_y + \hat{A}_y)^2\right]. \tag{16}$$

In the Abelian case $[\hat{A}_x, \hat{A}_y] = 0$. We can therefore choose a common eigenbasis for both matrices: $\hat{A}_x/\hbar = \mathrm{diag}\{q_{1x}, q_{2x}\}$ and $\hat{A}_y/\hbar = \mathrm{diag}\{q_{1y}, q_{2y}\}$. As a consequence, we recover two independently displaced quadratic spectra $E_j(\vec{k}) = (\hbar^2/2m)(\vec{k} + \vec{q})^2$, where $\vec{q}_j = (q_{jx}, q_{jy})$.

In the non-Abelian case, on the other hand, we cannot simultaneously diagonalize both matrices, and as a consequence the spectrum becomes distorted. Let us consider a simple, but representative, case, namely $\hat{A}_x = q_x\hat{\sigma}_x$, $\hat{A}_y = q_y\hat{\sigma}_z$. Employing the Fourier-like transformation

$$\vec{\psi}(x, y) = \sum_{k_x,k_y} \mathrm{e}^{\mathrm{i}k_y y\hat{\sigma}_z} \left(\frac{1+\mathrm{i}\hat{\sigma}_y}{\sqrt{2}}\right) \mathrm{e}^{\mathrm{i}k_x x\hat{\sigma}_z}\vec{\phi}(k_x, k_y) \tag{17}$$

with $k_{x,y} = 2\pi n_{x,y}/L$, we may transform the time-independent Schrödinger equation $E\vec{\psi}(x, y) = \hat{H}\vec{\psi}(x, y)$ into

$$\frac{2mE}{\hbar^2}\vec{\phi}(k_x, k_y) = \left[k_x^2 + q_x^2 + (k_y + q_y)^2\right]\vec{\phi}(k_x, k_y) + 2q_x k_x\vec{\phi}(k_x, -k_y). \tag{18}$$

Diagonalizing the system of equations for $\phi(k_x, \pm k_y)$, we obtain two eigenenergies

$$\frac{2mE_\pm}{\hbar^2} = k_x^2 + q_x^2 + k_y^2 + q_y^2 \pm 2\sqrt{k_x^2 q_x^2 + k_y^2 q_y^2}. \tag{19}$$

Note that in the Abelian situation $q_x = 0$ (or $q_y = 0$), and, as expected, there is no coupling between momenta in different directions. However, due to the non-Abelian character, even for a constant gauge there is a nontrivial coupling between the different directions.

4. Landau-like non-Abelian gauge

4.1. Periodic boundary conditions

We consider in the following a matrix generalization of the Landau gauge, namely $\hat{A} = (\hbar\kappa\hat{M}_x, B_0\hat{M}_y x, 0)$ (the usual Landau gauge is of the form $(0, B_0 x, 0)$). We will assume that the matrices $\hat{M}_x$ and $\hat{M}_y$ are constant. Then the Hamiltonian of the 2D system becomes:

$$\hat{H} = \frac{1}{2m}\left[(\hat{p}_x + \hbar\kappa\hat{M}_x)^2 + (\hat{p}_y + B_0\hat{M}_y x)^2\right]. \tag{20}$$

We first discuss the typical textbook situation, in which the particles (which are assumed to be confined on the xy-plane) are considered as confined in a 2D box of side L with periodic boundary conditions (i.e. a toroidal configuration). We are particularly interested in how the non-Abelian character of the fields destroys the usual Landau-level structure of the energy eigenstates. In the following subsection we shall discuss a slightly different scenario closer to actual experimental conditions.

As in section 3, if $[\hat{M}_x, \hat{M}_y] = 0$, one can find a common eigenbasis $\{\vec{e}_1, \vec{e}_2\}$, such that in this basis $\hat{M}_x = \mathrm{diag}\{\gamma_1, \gamma_2\}$ and $\hat{M}_y = \mathrm{diag}\{\lambda_1, \lambda_2\}$, and hence the Hamiltonian is also diagonal in this basis. Since we assume periodic boundary conditions we can thus consider wavefunctions of the form

$$\vec{\psi}_j(\vec{r}) = \sum_{n_y} \mathrm{e}^{\mathrm{i}(2\pi/L)n_y y + \mathrm{i}\kappa\gamma_j q} v_j(n_y, x)\vec{e}_j, \tag{21}$$

such that

$$E v_j(q) = \left[\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_j^2 q^2\right] v_j(q). \tag{22}$$

where $q = x + (2\pi\hbar n_y/LB_0\lambda_j)$, $p = -\mathrm{i}\hbar\partial/\partial q$, and $\omega_j = B_0|\lambda_j|/m$ is the cyclotron frequency for the state j. Hence, for the Abelian case we obtain two different sets of Landau levels with energies $E_j(n) = \hbar\omega_j(n+1/2)$, and degeneracies $g_j = B_0\lambda_j L^2/2\pi\hbar$. Note that if $|\lambda_1| = |\lambda_2|$, as is the case for $M_y = \hat{\sigma}_z$, then the two sets of Landau levels are degenerate.

Let us now discuss what happens if on the contrary $[\hat{M}_x, \hat{M}_y] \neq 0$. We work (without lack of generality) in the basis in which $M_y = \hat{\sigma}_z$. Note that the ansatz

$$\vec{\psi}(\vec{r}) = \sum_{n_y} \mathrm{e}^{\mathrm{i}2\pi/Ln_y y\hat{\sigma}_z}\vec{u}(n_y, x), \tag{23}$$

also fulfills periodic boundary conditions. We insert this ansatz in the eigenvalue equation to obtain

$$E\vec{u}(n_y, x) = \left[\frac{\hat{\Pi}^2}{2m} + \frac{\hbar^2}{2m}\left(\frac{2\pi n_y}{L} + \frac{B_0}{\hbar}x\right)^2\right]\vec{u}(n_y, x) + \left[\hat{\sigma}_z[\hat{\Pi}^2, \hat{\sigma}_z]\right]\left[\frac{\vec{u}(n_y, x) - \vec{u}(-n_y, x)}{4m}\right], \tag{24}$$

where $\hat{\Pi} = \hat{p}_x + \hbar\kappa\hat{M}_x$. For the Abelian case, $[\hat{M}_x, \hat{\sigma}_z] = 0$, the last term vanishes, and we get the same equation as previously. However, for the non-Abelian case, the last term introduces a coupling between the modes with n_y and $-n_y$, and hence there is an explicit dependence on n_y. As a consequence of that, the degeneracy of the Landau levels is lifted.

For the particular case of $\hat{M}_x = \hat{\sigma}_y$, we get the following set of coupled equations ($\epsilon = E - \hbar^2\kappa^2/2m$):

$$\epsilon\vec{u}(n_y, x) = \left[\frac{\hat{p}_x^2}{2m} + \frac{B_0^2}{2m}\left(x + \frac{2\pi\hbar n_y}{B_0 L}\right)^2\right]\vec{u}(n_y, x) + \frac{\hbar\kappa}{m}\hat{p}_x\hat{\sigma}_y\vec{u}(-n_y, x), \tag{25}$$

$$\epsilon\vec{u}(-n_y, x) = \left[\frac{\hat{p}_x^2}{2m} + \frac{B_0^2}{2m}\left(x - \frac{2\pi\hbar n_y}{B_0 L}\right)^2\right]\vec{u}(-n_y, x) + \frac{\hbar\kappa}{m}\hat{p}_x\hat{\sigma}_y\vec{u}(n_y, x). \tag{26}$$

The coupling prevents the re-absorption of n_y in the definition of a new q variable, as was done in the Abelian case, and hence the spectrum explicitly depends on n_y. Note that we are imposing periodic boundary conditions, and hence x is in a ring of perimeter L. In this sense, $\pm L/2$ are the same point, and this must be taken into account when considering the harmonic oscillator potential in each equation.

Note that the previous equations involve the coupling of harmonic oscillator wavefunctions centered at $\pm x_\mathrm{c}(|n_y|)$, with $x_\mathrm{c}(|n_y|) = 2\pi\hbar|n_y|/B_0 L$. Hence, the smaller the overlapping between coupled wavefunctions, i.e. the larger x_c, the smaller the coupling, and as a consequence only sufficiently small values of n_y will be affected by the non-Abelian coupling. This point becomes clear after performing first-order perturbation theory assuming a small coupling κ. A straightforward calculation shows that the lowest Landau levels, which correspond to the lowest eigenvalues of each harmonic oscillator, experience a maximal energy shift

$$\frac{\Delta E}{\hbar\omega_\mathrm{c}} = (\kappa l_\mathrm{c})\frac{n_y}{\Delta n_y}\mathrm{e}^{-n_y^2/\Delta n_y^2}, \tag{27}$$

where $l_\mathrm{c}^2 = \hbar/m\omega_\mathrm{c}$ is the magnetic length, and $\Delta n_y = \sqrt{g/2\pi}$, with g the degeneracy of the unperturbed Landau levels. Note that for $n_y = 0$ the first correction should be quadratic in κ,

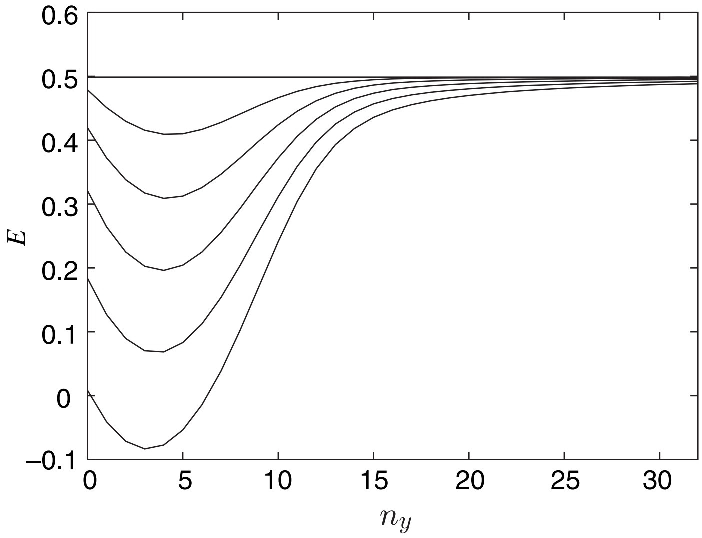

Figure 1. Lowest eigenvalue ϵ/ω_c as a function of n_y, for $g = 128$ and $\kappa l_c = 0, 0.2, 0.4, 0.6, 0.8, 1.0$ (from the uppermost to the lowermost curve).

whereas for $n_y \neq 0$ it should be linear. Clearly, the relative importance of the non-Abelian corrections should decrease as $1/\sqrt{g}$. In particular, the maximal energy shift $\langle \Delta E \rangle$ averaged over the different n_y can be approximated as $\langle \Delta E \rangle / \hbar\omega_c \simeq (\kappa l_c)/\sqrt{2\pi g}$.

We have solved numerically for the eigenvalues of equations (25) and (26) imposing periodic boundary conditions, for different values of g which controls the strength of the magnetic field applied, and κl_c which provides the strength of the non-Abelian corrections. The value of $L/l_c = \sqrt{2\pi g}$ is chosen in all simulations. Figure 1 shows the behavior of the lowest eigenvalue as a function of n_y for $g = 128$ and $\kappa l_c = 0, 0.2, 0.4, 0.6, 0.8, 1.0$ (from the uppermost to the lowermost curve). The figure follows approximately the perturbative result. For $n_y = 0$ a higher order contribution appears, but note that a quadratic law follows for small κ, and not a linear one, as in the case for $n_y \neq 0$. As expected from the previous calculations only values of n_y up to the order of $\sqrt{g}$ contribute significantly to the shift of the lowest Landau level.

Figure 2 shows the behavior of the Landau levels for $g = 128$, and $\kappa l_c = 0$ (left) and 0.6 (right). The figures are presented as histograms in intervals of $0.05\hbar\omega_c$, in order to reveal more clearly the destruction of the Landau levels. Note that the gaps (of energy $\hbar\omega_c$) between the Landau levels are filled, and the peaks in the density of states are progressively reduced. For sufficiently large κ the Landau level structure therefore disappears.

4.2. Absorbing boundary conditions

In the previous section, we discussed how the non-Abelian character of the gauge field significantly modifies the textbook Landau level structure. In the following, we consider a slightly different physical scenario which is closer to the actual experimental conditions discussed in section 2. The particular procedure devised for the generation of the non-Abelian Landau gauge demands that the x-coordinate cannot be considered as periodic. We take the same box configuration as for the previous subsection, but assume absorbing boundary conditions in

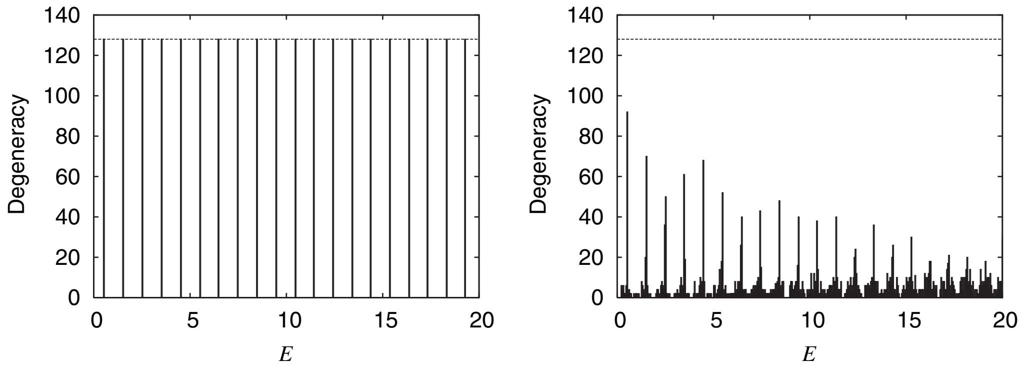

Figure 2. Landau level structure for periodic boundary conditions, $g = 128$, and $\kappa l_{\rm c} = 0$ (left) and $\kappa l_{\rm c} = 0.6$ (right). We employ (see text) $\hat{M}_x = \hat{\sigma}_y$ and $\hat{M}_y = \hat{\sigma}_z$.

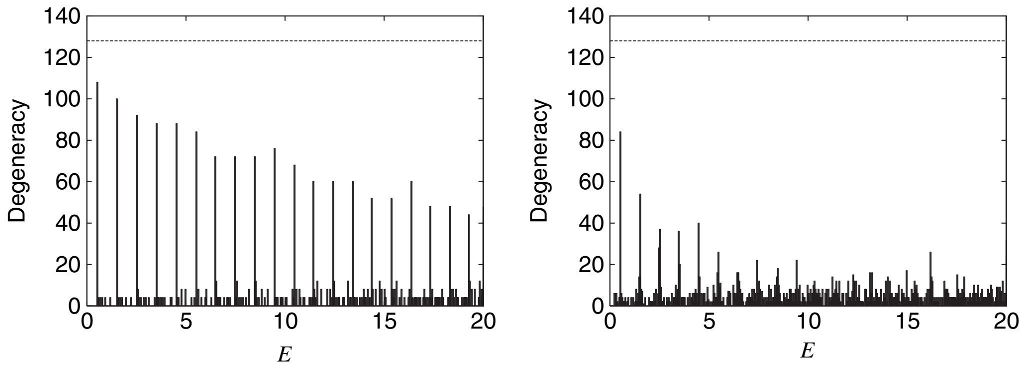

Figure 3. Same cases considered in figure 2 but with absorbing boundary conditions (see text).

the x-direction, while keeping for simplicity periodic boundary conditions in the y-direction. We consider exactly the same gauge discussed in the previous subsection. The spectrum is provided by equations (25) and (26) but imposing absorbing boundary conditions. Figure 3 shows the lowest Landau levels for the same cases discussed in figure 2.

Even for the Abelian case the Landau level structure is of course affected by the absorbing boundary conditions. In the Abelian case, as discussed in the previous section, the problem reduces to two decoupled equations for harmonic oscillators centered at $\pm x_{\rm c}(|n_y|)$. Clearly when $x_{\rm c}$ approaches L the levels of the resulting potential become greatly distorted, leading to a significant modification of the Landau level structure when n_y approaches g. This reduces the effective degeneracy of the lowest Landau levels to values smaller than g. The effective degeneracy, as shown in the figures, becomes smaller for higher Landau levels. The non-Abelian effect leads, as in the previous subsection, to the eventual destruction of the Landau level structure.

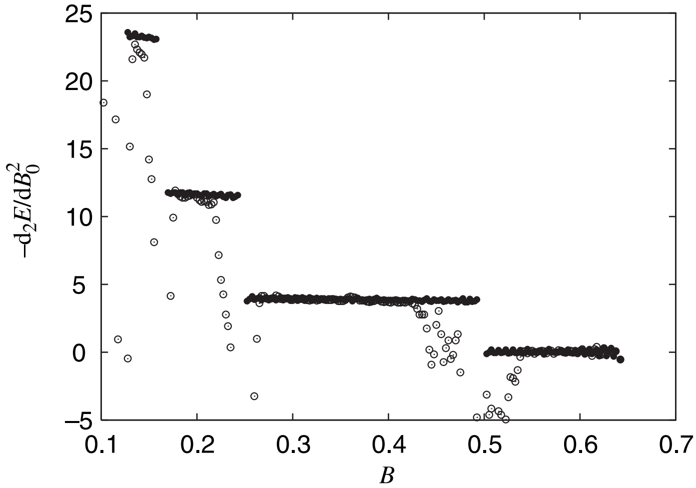

Figure 4. Value of $d^2\bar{E}/dB_0^2$ as a function of the applied magnetic field B_0, for the same cases discussed in figure 2, with $\kappa = 0$ (filled circles) and $\kappa = 0.6$ (hollow circles).

4.3. Modified de Haas–van Alphen effect

The destruction of the Landau level structure has experimentally relevant consequences for the behavior of cold atomic gases. As an example we can consider the case of an ideal two-component Fermi gas under the previously mentioned non-Abelian gauge potential (we consider a temperature $T \ll T_F$, where T_F is the Fermi temperature). Equivalent to the well-known de Haas–van Alphen effect [23], we may study the energy per particle, $\bar{E} = E/N$, of the Fermi gas, as a function of the applied magnetic field B_0, or equivalently of g. This energy may be monitored by measuring the released energy in time-of-flight experiments. For $\kappa = 0$ (Abelian case) $d^2\bar{E}(B_0)/dB_0^2$ presents a typical configuration of plateaux, due to the degeneracy of the Landau levels. The destruction of the Landau level structure significantly distorts this picture, rounding-off or eventually destroying this plateaux configuration (see figure 4).

5. Symmetric gauge

In this section, we consider an ideal cold atomic sample in an isotropic harmonic trap of frequency ω, in the presence of a non-Abelian generalization of the symmetric gauge of the form $\hat{A} = \hat{A}_\rho\hat{\rho} + \rho\hat{A}_\varphi\hat{\varphi}$. Although the tripod scheme is not suitable for the experimental realization of this gauge, we include the analysis of this gauge field for completeness of our discussion. Other ways of generating non-Abelian gauge fields, such as lattice techniques [15] should be employed in this case. In the following we consider $A_\rho = \hbar\kappa\hat{U}_\rho$, $A_\varphi = B_0\hat{U}_\varphi$, where $\hat{U}_{\rho,\varphi}$ are linear combinations of $\{\hat{1}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$.

The corresponding time-independent Schrödinger equation is of the form

$$E\vec{\psi} = \frac{1}{2m}\left[-i\hbar\vec{\nabla} + \hat{A}\right]^2\vec{\psi} + \frac{m\omega^2}{2}\rho^2\vec{\psi}. \tag{28}$$

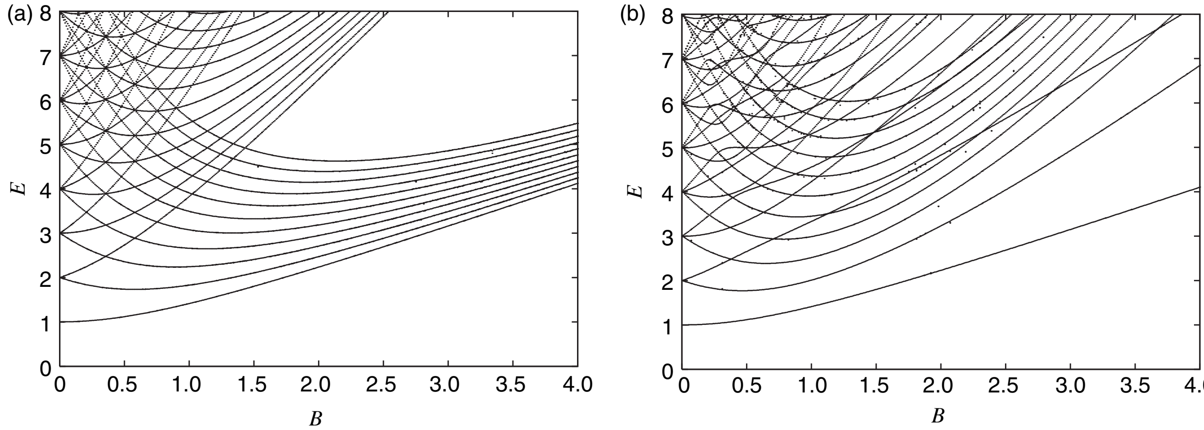

Figure 5. Fock–Darwin spectrum $E/\hbar\omega$ for the Abelian (left) and non-Abelian (right) cases discussed in the text as a function of $b_0 = \omega_c/\omega$.

Performing the gauge transformation $\vec{\psi} = \exp[-i\hat{A}_\rho \rho/\hbar]\vec{\phi}$, the Schrödinger equation transforms into

$$E\vec{\psi} = \frac{1}{2m}\left[-i\hbar\vec{\nabla} + \hat{\varphi}C_\varphi(\rho)\rho\right]^2 \vec{\psi} + \frac{m\omega^2}{2}\rho^2\vec{\psi}, \tag{29}$$

where

$$C_\varphi(\rho) = e^{i\hat{A}_\rho \rho/\hbar}\hat{A}_\varphi e^{-i\hat{A}_\rho \rho/\hbar}. \tag{30}$$

Note that C_φ becomes ρ dependent and different from $\hat{A}_\varphi$ if $[\hat{A}_\rho, \hat{A}_\varphi] \neq 0$.

If we now consider the solutions with angular momentum l, $\vec{\phi} = \vec{R}_l \rho^{|l|} e^{il\varphi}$, we obtain

$$E\vec{R}_l = -\frac{1}{2}\left[\frac{d^2}{d\rho^2}\vec{R}_l + \frac{(2|l|+1)}{\rho}\frac{d}{d\rho}\vec{R}_l\right] + \frac{1}{2}\left[1 + C_\varphi(\rho)^2\right]\rho^2\vec{R}_l + l\hat{C}_\varphi(\rho)\vec{R}_l, \tag{31}$$

where we reduce the equations to a dimensionless form by employing oscillator units for the energy ($\hbar\omega$) and for the length ($l_{ho} = \sqrt{\hbar/m\omega}$). In the previous equation $\hat{C}_\varphi(\rho) \equiv (\omega_c/\omega)\exp[i\kappa\hat{U}_\rho\rho]\hat{U}_\varphi\exp[-i\kappa\hat{U}_\rho\rho]$, where $\omega_c = B_0/m$ is the corresponding cyclotron frequency.

As mentioned above, the non-Abelian character of the gauge field induces an additional ρ-dependent potential. It severely distorts the standard Fock–Darwin spectrum which is expected for the Landau-level structure in the presence of a symmetric gauge and a harmonic potential, as shown in figure 5. An inspection of the level structure shows that not only are the eigenenergies modified, but also the ordering of the different eigenstates becomes distorted as a consequence of the non-Abelian potential. As a consequence of this extra ρ-dependent potential, an ideal Fermi gas at zero temperature shows a significantly distorted density profile in the presence of the non-Abelian gauge field, as shown in figure 6.

6. Conclusions

In this paper, we have analyzed the physics of ultracold gases in the presence of a non-Abelian gauge field. We have first studied how different types of non-Abelian fields may be created by means of relatively simple laser arrangements with atoms described by an electronic tripod

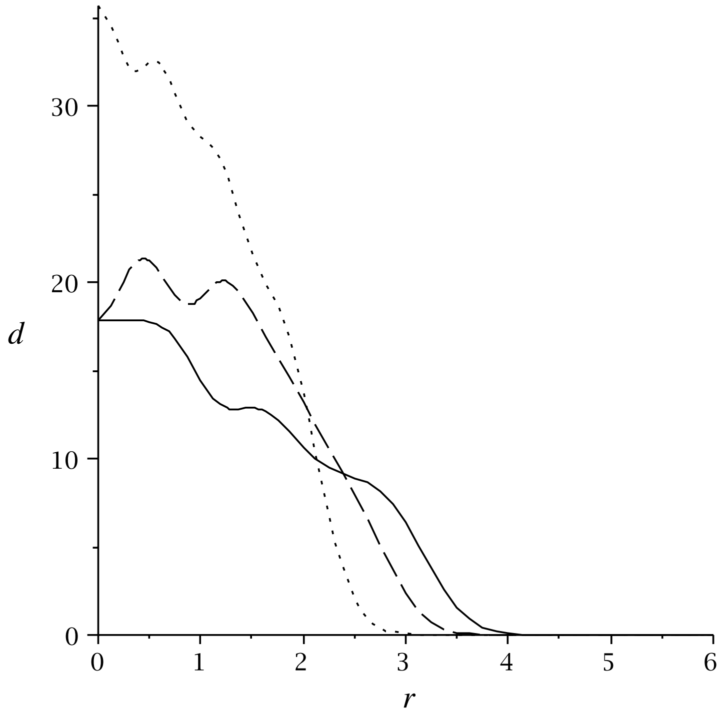

Figure 6. Comparison between the density profile for an ideal Fermi gas occupying up to 56 eigenlevels at zero temperature for the Abelian (solid) and non-Abelian cases discussed in the text with different values of $\kappa = 1$ (dashed) and $\kappa = 5$ (dotted).

level scheme, including a non-Abelian generalization of the Landau gauge. In a second part, we have considered the nontrivial effects that the non-Abelian character has on the eigenlevel structure of the cold atomic system. In particular, we have shown that exclusively due to the non-Abelian character of the field, the usual Landau level structure is severely distorted, and even eventually destroyed. We have shown that this effect may be observable in an equivalent experiment to the well-known de Haas–van Alphen effect. The distortion of the Landau levels leads to a significant modification of the usual plateaux-like signal characteristic for the de Haas–van Alphen effect. Finally, we have completed our analysis of a non-Abelian version of the symmetric gauge. We have shown that the Fock–Darwin spectrum is significantly distorted in the presence of non-Abelian fields, due to the presence of an extra potential, which is a purely non-Abelian effect.

Acknowledgments

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References

[1] Jackson J D 2002 *Classical Electrodynamics* (New York: Wiley)
[2] Mead C A 1992 *Rev. Mod. Phys.* **64** 51
[3] Lavrentovich O D 1986 *JETP Lett.* **43** 382
Chernodub M N 2006 *JETP Lett.* **83** 268
[4] Pitaevskii L P and Stringari S 2004 *Bose–Einstein Condensation* (Oxford: Clarendon)
[5] Kavoulakis G M, Mottelson B and Pethick C J 2000 *Phys. Rev.* A **62** 063605
[6] Schweikhard V, Coddington I, Engels P, Mogendorff V P and Cornell E A 2004 *Phys. Rev. Lett.* **92** 040404
[7] Bretin V, Stock S, Seurin Y and Dalibard J 2004 *Phys. Rev. Lett.* **92** 050403
[8] Wilkin N K and Gunn J M F 2000 *Phys. Rev. Lett.* **84** 6
[9] Wilczek F and Zee A 1984 *Phys. Rev. Lett.* **52** 2111
[10] Juzeliūnas G and Öhberg P 2004 *Phys. Rev. Lett.* **93** 033602
[11] Juzeliūnas G, Öhberg P, Ruseckas J and Klein A 2005 *Phys. Rev.* A **71** 053614
[12] Ruseckas J, Juzeliūnas G, Öhberg P and Fleischhauer M 2005 *Phys. Rev. Lett.* **95** 010404
[13] Juzeliūnas G, Ruseckas J, Lindberg M, Santos L and Öhberg P 2008 *Phys. Rev.* A **77** 011802
[14] Jaksch D and Zoller P 2003 *New J. Phys.* **5** 56
[15] Osterloh K, Baig M, Santos L, Zoller P and Lewenstein M 2005 *Phys. Rev. Lett.* **95** 010403
[16] Mueller E J 2004 *Phys. Rev.* A **70** 041603
[17] Sørensen A S, Demler E and Lukin M D 2005 *Phys. Rev. Lett.* **94** 086803
[18] Satija I I, Dakin D C and Clark C W 2006 *Phys. Rev. Lett.* **97** 216401
[19] Unanyan R G, Fleischhauer M, Shore B E and Bergmann K 1998 *Opt. Commun.* **155** 144
[20] Unanyan R G, Shore B W and Bergmann K 1999 *Phys. Rev.* A **59** 2910
[21] McGloin D, Spalding G C, Melville H, Sibbett W and Dholakia K 2003 *Opt. Express* **11** 158
[22] Whyte G and Courtial J 2005 *New J. Phys.* **7** 117
[23] Landau L D and Lifshitz E M 2002 *Statistical Physics* Part 1 (Oxford: Butterworth-Heinemann)