MULTIPLICATIVE STOCHASTIC SEQUENCE OF EVENTS AS A MODEL OF POWER-LAW NOISE

V. GONTIS AND B. KAULAKYS

Institute of Theoretical Physics and Astronomy, Vilnius University, A. Gostauto 12, 2600 Vilnius, Lithuania, gontis@ktl.mii.lt

We introduce the stochastic multiplicative model of time intervals between the events, defining multiplicative point process and analyze the statistical properties of the signal. Such a model system exhibits power-law spectral density $S(f)\sim 1/f^{\beta}$, scaled as power of frequency for various values of β, including β=1/2, 1 and 3/2. Explicit expressions for the power spectrum in the low frequency limit and for the distribution density of the interevent time are obtained. The counting statistics of the events is analyzed as well. The specific interest of our analysis is related with the financial markets, where long-range correlations of price fluctuations largely depend on the number of transactions.

1. Introduction

Stochastic point processes may be used for description of phenomena that occur as random sequences of events. Considerable part of such systems in physics, biomedicine, geophysics and economics are fractal as their statistics exhibit scaling. The scaling leads to the power-law dependencies of the scaled quantities [1]. The aim of this contribution is to introduce the multiplicative stochastic model of the time interval between events in stochastic sequence defining the multiplicative point process. The model of $1/f$ noise based on the Brownian motion of the time interval between subsequent pulses proposed in Refs. [2,3] has been adopted for the reproduction of the spectral properties of trading activity in financial markets [4]. From the central limit theorem it follows that the simple additive Brownian model of the time interval between events should lead to the Gaussian probability distribution of the time interval. Therefore, we introduce the stochastic multiplicative model for the interevent time, defining the multiplicative point process [5]. The model exhibits the first and the second order power law statistics and serves as the theoretical description of the empirical financial time series [6]. Specific interest of our analysis is a relation between the origin of the power-law distributions and power-law correlations in the financial time series. The model with the adjusted parameters reproduces the power spectra of trading activity and the exponent of the power-law probability distribution of the trading activity observed in the financial markets. [6]. Apparently, the multiplicative point process can be useful for the modeling of a wide variety of natural systems as well as of the processes in economics and finance.

2. Multiplicative point process

We consider a signal $I(t)$ as a sequence of the random correlated pulses

$$I(t) = \sum_k a_k d(t - t_k) \tag{1}$$

where a_k is a contribution to the signal of one pulse at the time moment t_k, for example, a contribution of one transaction to the financial data. When a_k is a constant, the process (1) is completely defined by the set of events $\{t_k\}$ or equivalently by the set of interevent intervals $\{t_k = t - t_k\}$. Kaulakys and Meskauskas [2-3] showed analytically that the relatively slow Brownian fluctuations of the interevent time t_k exhibited $1/f$ fluctuations of the signal $I(t)$. Power spectral density of the signal (1) can be written as

$$S(f) = \lim_{T\to\infty}\left\langle \frac{2a^2}{T} \sum_{k=k\,\mathrm{min}}^{k\,\mathrm{max}} \sum_{q=k\,\mathrm{min}-k}^{k\mathrm{max}-k} \exp(-i2p\, f\, \Delta(k;q)) \right\rangle \tag{2}$$

where T is the observation time, $\Delta(k;q) = t_{k+q} - t_k$ is the difference of pulses occurrence times t_{k+q} and t_k, a denotes expectation of a_k, while $k_{\min}$ and $k_{\max}$ are minimal and maximal values of index k in the time interval of observation T.

We will study the multiplicative processes defined by the stochastic iterative equation

$$t_{k+1} = t_k + g t_k^{\,2m-1} + t_k^{\,m} s e_k. \tag{3}$$

Here the interevent time t_k fluctuates due to the external random perturbation by a sequence of uncorrelated normally distributed random variable e_k with a zero expectation and unit variance, s denotes the standard deviation of the white noise and $g \ll 1$ is a damping constant. The diffusion described by Eq. (3) has to be restricted in some time interval $t_{\min} < t < t_{\max}$.

Pure multiplicativity corresponds to the parameter $m = 1$. Nevertheless, other values of m can produce power laws, as well, and the explicit expressions can be derived without the loss of generality. The iterative relation (3) can be rewritten as a continuous Langevine stochastic differential equation in k space

$$\frac{dt}{dk} = g t_k^{\,2m-1} + t_k^{\,m} s x_k \tag{4}$$

where $\langle x_k x_{k'} \rangle = d(k - k')$. The stationary solution of the corresponding Fokker-Plank equation with a zero flow gives the long time probability distribution of t in the space k

$$P_k(t) = C_t t^{a} \tag{5}$$

where C_t has to be defined from the normalization $\int_{t\,\min}^{t\,\max} P(t)dt = 1$ and $a = \frac{2g}{s^2} - 2m$.

3. Power spectral density and counting statistics

The power spectral density is a well-established measure of long-time correlations and is widely used in stochastic systems. For the normal distribution of $\Delta(k;q)$ Eq. (2) takes the form

$$S(f)=\lim_{T\to\infty}\frac{2a^2}{T}\sum_{k,q}\exp\{i2p\,f\left\langle\Delta(k;q)\right\rangle-2p^2f^2s_{\Delta}^2(k;q)\}\tag{6}$$

where $\Delta(k;q)$ can be expressed from the solution of the multiplicative stochastic equation (4),

$$\Delta(k;q)=\sum_{i=1}^{q}t_k\exp((g-\frac{1}{2}s^2)t^{2m-2}i+\sum_{j=1}^{i}s\,t^{m-1}e_j)\,.\tag{7}$$

Averaging over the normal distributions of e_j yields the explicit expressions for the mean $\left\langle\Delta(k;q)\right\rangle$ and variance $s_{\Delta}^2(k;q)$,

$$\left\langle\Delta(k;q)\right\rangle=t_k q+\frac{g}{2}t_k^{2m-1}q^2+o(g^2)\,,\tag{8}$$

$$s_{\Delta}^2(k;q)=t_k^{2m}\frac{s^2}{3}q^3\,.\tag{9}$$

In the low frequency limit Eqs. (6) and (8) yield the power spectral density

$$S_m(f)=\frac{2C_t a^2}{\sqrt{p}\,\bar{t}(3-2m)f}(\frac{g}{p\,f})^{\frac{a}{3-2m}}\operatorname{Re}\int_{x_{\min}}^{x_{\max}}\exp(-i(x-\frac{p}{4}))erfc(\sqrt{-ix})x^{\frac{a}{3-2m}-\frac{1}{2}}dx.\tag{10}$$

Here we introduce the scaled variable $x=\frac{p\,f}{g}t^{3-2m}$ and $\bar{t}$ is the expectation of t_k. For $x_{\min}\to 0$ and $x_{\max}\to\infty$ Eq. (10) yields the explicit form of the power spectrum

$$S_m(f)=\frac{C_t a^2}{\sqrt{p}\,\bar{t}(3-2m)f}(\frac{g}{p\,f})^{\frac{a}{3-2m}}\frac{\Gamma(\frac{1}{2}+\frac{a}{3-2m})}{\cos(\frac{pa}{2(3-2m)})}\,.\tag{11}$$

Eq. (11) proves that the multiplicative point process exhibits a general model of signals with the power spectral density $S(f)\sim f^{-b}$. The scaling exponent is

$$b=1+\frac{2g/s^2-2m}{3-2m}\,.\tag{12}$$

We suppose that this stochastic model with the parameters resulting in $b\simeq 1$ can be adopted for a wide variety of real systems. First of all we assume that $a\equiv 1$ and the signal

$I(t)$ counts the transactions in financial markets. Then the number of transactions in selected time window t_d defined as $N(t) = \int_t^{t+t_d} I(t)dt$ measures the trading activity. For the pure multiplicative model, $m = 1$, Eqs. (7) and (8) define the relation between N and t. After substitution $t_k \to t$, $q \to N$ and $\langle \Delta(k;q) \rangle \to t_d$ we get

$$t N + \frac{g}{2} t\ N^2 = t_d. \tag{13}$$

This relationship may be used for definition of the probability density function of N in a real time from the relation $P_t(N)dN = P_t(t)dt$,

$$P_t(N) = \frac{C_t' t_d^{2+a} (1 + g N)}{N^{3+a} (1 + \frac{g}{2} N)^{3+a}} \simeq \left\{ \begin{array}{ll} \dfrac{C_t' t_d^{2+a}}{N^{3+a}}, & N \ll g^{-1}, \\ \dfrac{C_t' 2^{3+a} (\frac{t_d}{g})^{2+a}}{N^{5+2a}}, & N \gg g^{-1} \end{array} \right\}. \tag{14}$$

In the case of pure multiplicativity, $m = 1$, the model has only one parameter $2g / s^2$ defining the scaling of the power spectral density, the power-law distributions of interevent time and the counting number N.

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References

[1] S. Thurner, et al., *Analysis, synthesis, and estimation of fractal-rate stochastic point processes*, Fractals **5** (1997) 565-595.

[2] B. Kaulakys. and T. Meskauskas, *Modeling 1/f noise*, Phys. Rev. E **58** (1998) 7013-7019.

[3] B. Kaulakys, *Autoregressive model of 1/f noise*, Phys. Lett. A **257** (1999) 37-42.

[4] V. Gontis, *Modeling share volume traded in financial markets*, Lithuanian J. Phys. **41** (2001) 551-555; cond-mat/0201514.

[5] V. Gontis, *Multiplicative stochastic model of the time interval between trades in financial markets*, Nonl. Anal.: Model. Contr.**7** (2002) 43-54; cond-mat/0211317.

[6] V. Gontis and B. Kaulakys, *Multiplicative point process as a model of financial markets*, cond-mat/0303089