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Interplay between positive feedbacks in the generalized CEV process

St. Reimann [a,*], V. Gontis [b], M. Alaburda [b]

[a] *Department of Management, Technology and Economics, ETH Zurich, Kreuzplatz 5, CH-8032 Zurich, Switzerland*

[b] *Institute of Theoretical Physics and Astronomy of Vilnius University, A. Gostauto 12, LT-01108 Vilnius, Lithuania*

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ABSTRACT

The dynamics of the *generalized* CEV process $dX_t = aX_t^n dt + bX_t^m dW_t (gCEV)$ is due to an interplay of two feedback mechanisms: State-to-Drift and State-to-Diffusion, whose degrees are n and m respectively. We particularly show that the gCEV, in which both feedback mechanisms are POSITIVE, i.e. $n, m > 1$, admits a stationary probability distribution P provided that $n < 2m - 1$. In this case the stationary pdf asymptotically decays as a power law $P(x) \sim \frac{1}{x^{\mu}}$ with tail exponent $\mu = 2m > 2$. Furthermore the power spectral density obeys $S(f) \sim \frac{1}{f^{\beta}}$, where $\beta = 2 - \frac{1+\epsilon}{2(m-1)}$, $\epsilon > 0$. The tail behavior of the stationary pdf as well as of the power-spectral density thus are both independent of the drift feedback degree n but governed by the diffusion feedback degree m. Bursting behavior of the gCEV is investigated numerically. Burst intensity S and burst duration T are shown to be related by $S \sim T^2$.

The dynamics of the state X_t of a system which is open to a rapidly fluctuating environment can be described by the non-linear stochastic differential equation

$$dX_t = f(X_t)dt + g(X_t)dW_t, \tag{1}$$

W_t the standard Wiener process, under the assumption that (1) noise enters linearly, and (2) the White Noise approximation is valid, see Ref. [1]. The drift and the diffusion 'coefficients' depend on the recent state X_t and hence represent 'State-to-Drift' or 'State-to-Diffusion' feedbacks, respectively. The resulting dynamics, and consequently properties such as the stationary pdf of the gCEV, the spectral density, and burst statistics, are shown to be due to the interplay between these two feedback mechanisms.

The following (informal) argument shows that if both feedback mechanisms $f(X_t)$ and $g(X_t)$ have a particular functional relation to each other, given by

$$f(X_t) = \alpha g(X_t) g'(X_t) \tag{2}$$

then their interaction generates a power-law like stationary probability distribution – if it exists – in that

$$P(x) \sim \frac{1}{g(x)^{2(1-\alpha)}}. \tag{3}$$

(Here, and in the following, the notation $F(x) \sim f(x)$ means that the function $F(x) = f(x)$ for large x.) Note that the proportionality factor α enters the coefficient of the power-law tail. The process considered in Ref. [2], $dX_t = b^2\left(m - \frac{\lambda}{2}\right) X_t^{2m-1} dt + bX_t^m dW_t$, corresponds to $g(X) = X_t^m$ and $\alpha = 1 - \frac{\lambda}{2m}$ so that the stationary pdf decays as a power-law according to $P(x) \sim \frac{1}{x^{\lambda}}$.

* Corresponding author.
E-mail address: streimann@ethz.ch (St. Reimann).

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1. The generalized CEV process (gCEV)

In the following we consider a particular setting, which is that the drift and diffusion coefficients obey $f(X_t) = aX_t^n$ and $g(X_t) = bX_t^m$. In this case one obtains the Ito diffusion process with positive drift and diffusion parameters a, b given by

$$\mathrm{d}X_t = aX_t^n \mathrm{d}t + bX_t^m \mathrm{d}W_t, \quad n, m > 0. \tag{4}$$

This process is a generalization of the Constant – Elasticity – of – Variance model, $\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t^m\mathrm{d}W_t$, which was originally proposed by Cox, Ingersoll and Ross to describe the dynamics of interest rates in an equilibrium economy and which plays an important role in Mathematical Finance, see references below. The gCEV process, Eq. (4), describes dynamics by the superposition of *two* different feedback scenarios: One is the *State-to-Drift feedback* incorporated in the deterministic part of the gCEV

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = aX_t^n, \tag{5}$$

while the other one concerns *State-to-Diffusion feedback* due to

$$\mathrm{d}X_t = bX_t^m \mathrm{d}W_t. \tag{6}$$

In the following, we will focus on the case where both dynamical components exhibits *positive feedback* simultaneously, in that we require

$$1 < n, \qquad m < \infty. \tag{7}$$

In this case gCEV dynamics results from the interplay of two positive feedback scenarios, each of which generates self-amplifiction, i.e. 'explosive' behavior in itself. This is easily seen in that the drift term with positive feedback gives rise to a Finite-Time-Singularity, i.e. $X_t \to \infty$ as $t \to t_c$, where $t_c = \frac{1}{n-1}X_0^{1-n}$, while the solution on the finite interval $[0, t_c]$ is $X_t \propto \frac{1}{(t_c-t)^{\frac{1}{n-1}}}$. Positive feedback in the state-to-diffusion term also leads to bursting behavior, in that X_t can attain arbitrary large values while it always remains finite. This follows from the fact that the solution of Eq. (6) is the inverse power of a d-dimensional Bessel process $X_t \propto \frac{1}{\|\mathbf{B}\|_2^{\frac{1}{m-1}}}$, where $\mathbf{B}$ is a d-dimensional Brownian motion, where $d = \frac{2m-1}{m-1} > 2$, for details see Ref. [3]. Since therefore $d(m) > 2$, it follows from the transitivity of $\mathbf{B}_t$, that $\mathbf{B}_t$ escapes to ∞ for $t \to \infty$ slower than t a.s., see Ref. [4], while the origin 0 is polar, i.e. it will not be touched by $\mathbf{B}_t$. On the other hand $\mathbf{B}_t$ has a positive probability to visit any finite ϵ-neighborhood of the origin before escaping. Consequently the dynamics exhibits arbitrary high but finite excursions.

Hence, for $n, m > 1$, both singularities are entirely different: While positive feedback in the state-to-drift component leads to a 'real' Finite-Time-Singularity in that $X_t \to \infty$ within $[0, t_c]$, X_t remains finite even if feedback in state-to-diffusion is positive. Nonetheless, if both positive feedbacks play in concert, the process exhibits a fat-tailed stationary probability distribution, provided that the State-to-Diffusion feedback is positive ($m > 1$) and strong enough with respect to the State-to-Drift feedback, i.e. $m > \frac{1}{2}(n+1)$. Bursting behavior is reflected in that the stationary pdf decays as a power law $P(x) \sim \frac{1}{x^\mu}$ for large x, with an exponent obeying $\mu = 2m > 2$. For large x, the tail exponent only depends on the *state-to-diffusion* feedback parameter m, while the *state-to-drift* feedback parameter n determines the pdf only for small x.

2. Stationary pdf of a generalized CEV process

2.1. The CEV process

The standard CEV process is obtained from Eq. (4) for $n = 1$

$$\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t^m\mathrm{d}W_t, \quad a, b > 0. \tag{8}$$

A typical time series generated by the CEV process for $m = \frac{3}{2}$ is shown in Fig. 1. An extensive discussion of the CEV process and its relation to other processes, including Bessel processes, can be found in Ref. [5,4]. While most prior research on the CEV process has been restricted to the case $0 < m < 1$, we instead focus on the case that state-to-diffusion is subject to positive feedback $m > 1$. A detailed discussion of this process and the following theorem, as well its proof, can be found in Ref. [3]. As shown there, a CEV process with $m > 1$ is equivalent to a radial Ornstein–Uhlenbeck process for order $\nu(m) = \frac{1}{2(m-1)}$ and hence admits a closed form analytical solution given below:

Result 1 (*Solution of the CEV Process for $m > 1$*). *The unique and strong solution of the CIR-CEV model, Eq. (8), with $m > 1$ is*

$$X(t) = c(m)\frac{1}{\|\mathbf{M}(t)\|_2^{1/(m-1)}}, \tag{9}$$

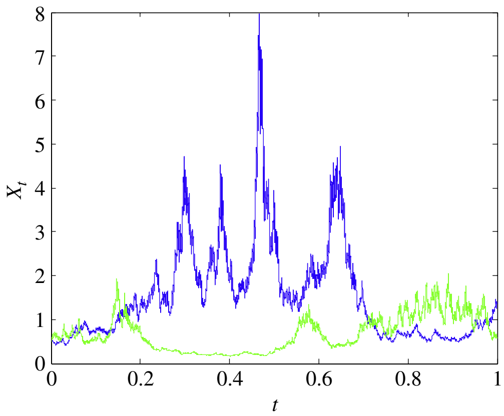

Fig. 1. Time trail of the CEV process $dX = aX_t^{\frac{4}{3}} + bX_t^{\frac{3}{2}} dW_t$ with $a = b = 1$. Number of iterations is 50.000.

with $c(m) = \Big(b(m-1)\Big)^{\frac{1}{1-m}}$*, where* $\mathbf{M}(t)$ *is a d-dimensional mean-reverting Ornstein–Uhlenbeck process, whose dimension is a function of the feedback parameter m given by*

$$\delta(m) = 2 + \frac{1}{m-1} \geq 2. \tag{10}$$

Its components obey $dM_i(t) = -\mu M_i dt + dB_i(t)$ *with* $a \geq 0$ *and* $B(t)$ *the standard Wiener process, while its square norm is* $\|\mathbf{M}(t)\|_2$.

The proof is based on the observation that the Lamberti transform of this process (8) takes the form of a radial Ornstein Uhlenbeck process of order $\nu(m) = \frac{1}{2(m-1)}$, see Ref. [3]. In this note we show that the CEV process Eq. (8) with positive *state-to-diffusion* feedback ($m > 1$) admits a stationary probability distribution, which is uni-modal and asymptotically decays as a power-law with its tail exponent proportional to m only.

Result 2 (*Stationary Pdf for the CEV Process for* $m > 1$). *Let* $dX_t = aX_t dt + bX_t^m dW_t$ *[CEV] be defined on the non-negative reals* $[0, \infty)$ *with* $a, b > 0$ *and* W_t *the Standard Wiener process. Then, if* $m > 1$ *a stationary probability distribution exists and is similar (not equal) to a Type-2 Gumbel distribution*

$$P(x) = \mathcal{N} x^{-2m} e^{-cx^{-2(m-1)}}, \tag{11}$$

where $c = \frac{2a}{b^2}\frac{1}{2(m-1)} > 0$ *and* $\mathcal{N} = \frac{2(m-1)}{c^{-\mu}\Gamma(\mu)} < \infty$ *is a normalization constant with* $\gamma = \frac{2m-1}{2(m-1)} > 1$. *The stationary pdf takes its unique maximum in* $x_* = \left(\frac{b^2}{a}\frac{m}{m-1}\right)^{-\nu(m)}$, *where* $\nu(m) = \frac{1}{2(m-1)}$ *is the index of radial Ornstein Uhlenbeck process equivalent to Eq. (8).*

Proof. As shown in Ref. [3] the solution of the CEV process [CEV] is an inverse power of a radial Ornstein Uhlenbeck process (rOU) of dimension $d = \frac{2m-1}{m-1} > 2$ for $1 < m < \infty$, given by $X_t \propto \frac{1}{\|\mathbf{M}\|_2^{\frac{1}{m-1}}}$, whose components obey $dM_i = -aM_i dt + dW_i$. Since $0 < \|\mathbf{M}\|_2 < \infty$, 0 and ∞ are natural boundaries for the CEV process with $m > 1$, thus the probability current over these boundaries is zero. □

For $m = \frac{3}{2}$ the stationary probability distribution asymptotically decays as a power law $P(x) \sim \frac{1}{x^3}$ for large x, more precisely

$$P(x) = \frac{2a}{b^2}\frac{1}{x^3} e^{-\frac{2a}{b^2}\frac{1}{x}} \sim \frac{1}{x^3}. \tag{12}$$

The pdf is shown in Fig. 3.

The equivalence between the CEV model with $m > 1$ and the rOU process of index $\nu(m) = \frac{1}{2(m-1)}$ also shows up in the functional form of the stationary pdf. In fact the distribution can be rewritten as

$$P(x) \propto x^{-(2+1/\nu(m))} \exp\left[-\nu(m) x^{-\frac{1}{\nu(m)}\frac{1}{x}}\right]$$

which shows that the exponential part of the pdf is controlled by the order of the corresponding rOU process. Furthermore, the stationary probability distribution belonging to the CEV process with $m > 1$ is uni-modal for all b and decays for large x as a power-law, i.e. $P(x) \sim \frac{1}{x^{2m}}$. Its graph is sketched in Fig. 2. As an immediate consequence we have

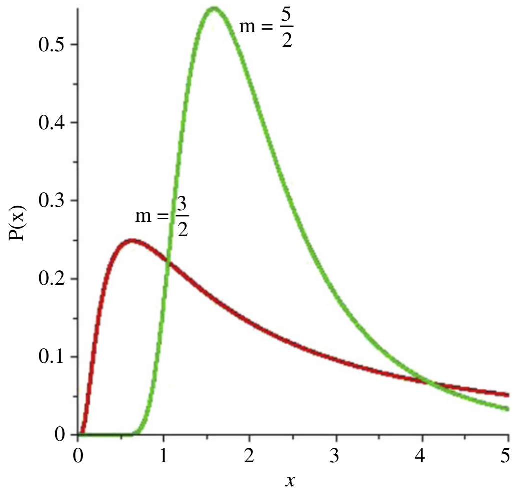

Fig. 2. The stationary pdf belonging to the Ito process $\mathrm{d}X = aX\mathrm{d}t + bX^m\mathrm{d}W$ for $m = 3/2$ and $m = \frac{5}{2}$, see Eq. (11), defined on the non-negative reals.

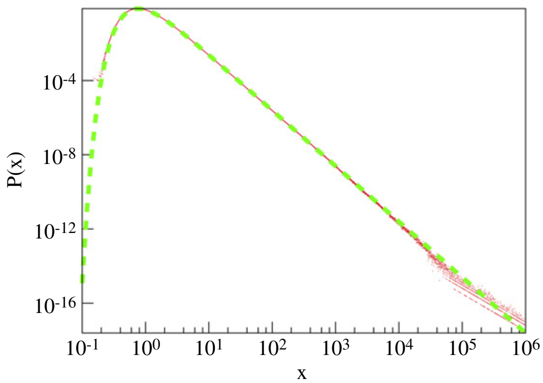

Fig. 3. The stationary pdf belonging to the CEV process $\mathrm{d}X = aX\mathrm{d}t + bX^{\frac{3}{2}}\mathrm{d}W$, with POSITIVE FEEDBACK on 'state-to-diffusion'. Simulated data are in red, while the green dotted line is the distribution due to Eq. (12). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Result 3. *Given the CEV process with feedback parameter $m > 1$ and let $\nu(m) = \frac{1}{2(m-1)}$. Then*

$$S(f) \sim \frac{1}{f^{\beta}}, \quad \beta(m) = 2 - (1+\epsilon)\nu(m), \tag{13}$$

where $\epsilon > 0$ is a small constant. Consequently $-\infty < \beta(m) < 2$ for $1 < m < \infty$.

This follows from the more general case, see below. Note that for $m = 3/2$ and $c = 0$, $\beta = 1$, so that in this case $S(f) \sim \frac{1}{f}$, see Fig. 4.

2.2. The generalized CEV process with positive feedback $n, m > 1$

The interplay between state-to-price feedback and state-to-diffusion becomes obvious when considering the Fokker–Planck equation belonging to the gCEV process. Note that by transforming X_t into $Y_t = \frac{1}{b(m-1)}\frac{1}{X^{m-1}}$, the gCEV process becomes $\mathrm{d}Y_t = \tilde{a}(Y_t) + \mathrm{d}W_t$ with drift

$$\tilde{a}(Y_t) = \frac{1}{b}\left[a\Big(b(m-1)Y_t\Big)^{\frac{n-m}{1-m}} - \frac{m}{m-1}\frac{1}{Y_t}\right]. \tag{14}$$

The corresponding potential $U(y) = \int^y \tilde{a}(y')\mathrm{d}y'$ of the corresponding Fokker–Planck equation is of the form

$$U(y) \propto \frac{1}{\eta}y^{\eta} - \ln y$$

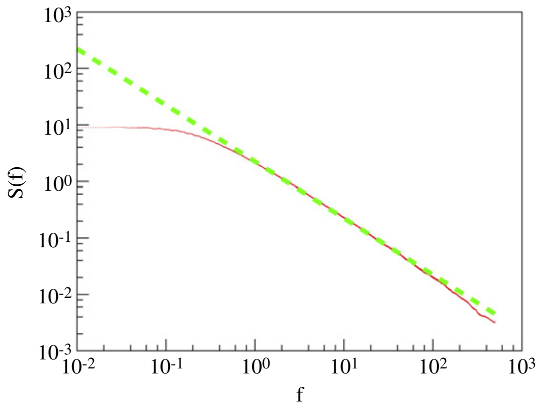

Fig. 4. The power spectral density of the CEV process $dX = aXdt + bX^{\frac{3}{2}}dW$. Simulated density in red, while the green dashed lines is the $1/f$ decay.

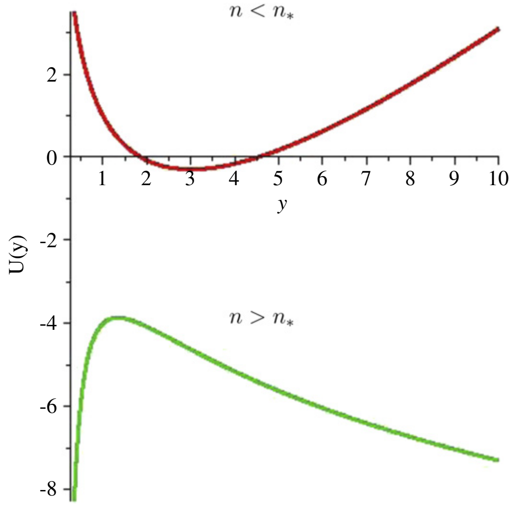

Fig. 5. Potential $U(y)$ of the Lamperti transformed gCEV process; the upper curve corresponds to $n < n_*$, while the lower one is for $n > n_*$.

where $\eta = \frac{n-(2m-1)}{1-m} \geq 0$ if $n \leq 2m - 1$ and positive otherwise. Thus there is a bifurcation occurring at $n_* = 2m - 1$, such that $U(y)$ is convex for $n < n^*$ and concave for $n > n*$, see Fig. 5. Particularly, for $n < n_*$, 0 and ∞ are repelling and the potential has a unique minimum. On the other hand, for $n > n_*$, both 0 and ∞ are attracting, while $U(y) \sim -\frac{1}{y^c} \to -\infty$ with some positive c for y approaching 0. In terms of X this means that for $n > n_*$, that small $X_t \to 0$, while large $X_t \to \infty$. For illustration, Figs. 1 and 6 show cGEV trails for $n = \frac{4}{3}$ and $n = \frac{5}{2}$, while in both cases $m = \frac{3}{2}$. Due to the $\wedge$-shape of the Fokker–Planck potential for $n = \frac{5}{2}$, X_t either tends to 0 or does diverge, see Fig. 6. Simulation is done for 15.000 iteration steps.

While for $n < n_*$ the gCEV process can be regarded as a diffusion trapped in a convex potential, i.e. behaves locally similar to an Ornstein–Uhlenbeck process, one can expect that in this case it admits a stationary probability density.

Result 4. *The generalized CEV process admits a stationary probability distribution $P(x)$ if $m > 1$ and $n < 2m-1$. The stationary probability distribution yields $P(x) = \mathcal{N} \frac{1}{x^{2m}} \exp(-cx^{-(2m-n-1)})$, where $c = \frac{2a}{b^2} \frac{1}{2m-n-1} > 0$, with normalization constant $\mathcal{N} = \frac{2m+1-n}{c^{-\mu}\Gamma(\mu)}$, $\mu = \frac{2m-1}{(2m-1)-n}$ and thus asymptotically decays as a power law with tail exponent $\mu > 2$*

$$P(x) \sim \frac{1}{x^{\mu}}, \quad \mu > 2.$$

The proof is a straightforward calculation.

The stationary probability density of the corresponding Fokker–Planck equation, see Ref. [6], is

$$P_0(x) = \frac{N}{b^2 x^{2m}} \exp\left(2 \int^x \frac{ax^n}{b^2 x^{2m}} dx\right)$$

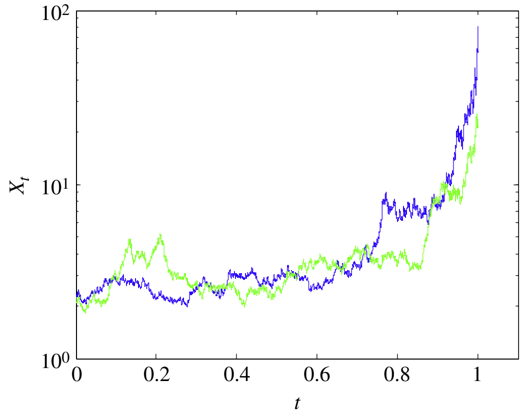

Fig. 6. Two realizations of the gCEV process both for $m = \frac{3}{2}$ and $n = \frac{5}{2} > n_*$, while $a = b = 1$. Number of iterations is 15.000.

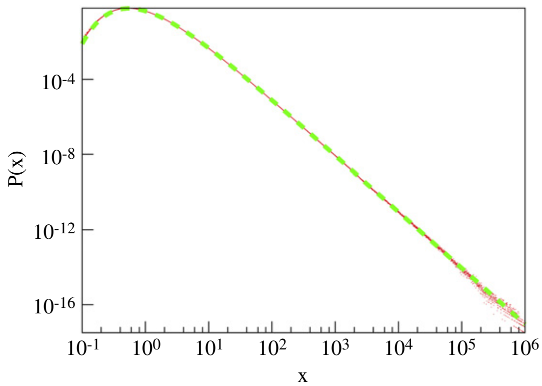

Fig. 7. The stationary pdf belonging to the gCEV process $\mathrm{d}X = aX^{\frac{4}{3}}\mathrm{d}t + bX^{\frac{3}{2}}\mathrm{d}W$, with POSITIVE FEEDBACK on both 'state-to-drift' and 'state-to-diffusion'.

$$= \frac{N}{x^{2m}} \exp\left(2\frac{a}{b^2}\int^x x^{n-2m}\mathrm{d}x\right)$$

$$= \begin{cases} \dfrac{N}{x^{2m}} \exp\left(\dfrac{2a}{b^2}\dfrac{x^{n+1-2m}}{n+1-2m}\right), & n-2m \neq -1 \\ Nx^{\frac{2a}{b^2}-2m}, & n = 2m-1. \end{cases} \tag{15}$$

In the case $n < 2m-1$ one can easily integrate $P_0(x)$ to get normalization constant $\mathcal{N}$. Note that this excludes the Geometric Brownian Motion case $[n = m = 1]$, while it includes the case that the process exhibits positive feedback of both: state-to-drift feedback $[n > 1]$ as well as state-to-diffusion $[m > 1]$. In fact, given the degree $n > 1$ of positive state-to-drift feedback, then the degree of state-to-diffusion feedback m must be positive as well while sufficiently large $m > \frac{1}{2}(n-1)$.

Fig. 7 shows the simulated distribution P (red) compared to the analytical solution, see Eq. (11), (green dashed lines) for the case $m = 3/2$ and $n = 4/3$, i.e. for the gCEV process in which both partial processes, Eqs. (5) and (6), exhibit positive feedback.

The power spectrum density of a gCEV asymptotically decays as a poer-law with tail index β, which is a direct function of the index ν of the related Bessel process and is, particularly independent of the feedback exponent n of the drift term.

Result 5. *The gCVE process Eq. (8) with feedback parameters $m > 1$, $n < 2m-1$ admits a power spectrum*

$$S(f) \sim \frac{1}{f^{\beta}}, \quad \beta = 2 - (1+\epsilon)\nu(m), \tag{16}$$

where $\epsilon > 0$ and $\nu(m) = \frac{1}{2(m-1)}$ is the index of the radial Ornstein–Uhlenbeck process equivalent to the CEV process with $m > 1$ and $c > 0$ is a small parameter.

Proof. The proof follows from results in Ref. [2] by noting that the gCEV process for large X_t can be approximated by $\mathrm{d}X_t = c(X)X_t^{2m-1}\mathrm{d}t + bX_t^m\mathrm{d}W_t$, where for $n < 2m+1$ the coefficient $c(X) = \frac{a}{mb^2}X_t^{n-(2m+1)}$ is almost constant for large X_t

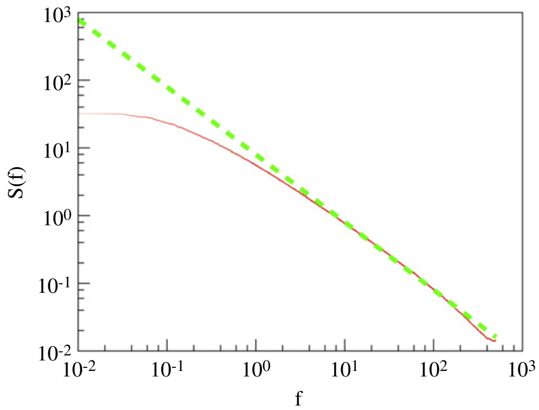

Fig. 8. The power spectral density $S(f) \sim \frac{1}{f^1}$ belonging to the gCEV process $\mathrm{d}X = aX^{\frac{4}{3}}\mathrm{d}t + bX^{\frac{3}{2}}\mathrm{d}W$.

and approaches 0 from above if $X_t \to \infty$. Therefore approximating the gCEV process for large but finite X by

$$\mathrm{d}X_t = cX_t^{2m-1}\mathrm{d}t + bX_t^m\mathrm{d}W_t, \quad c > 0 \tag{17}$$

we obtain Eq. 3 in Ref. [2] with the substitution $c = \left(m - \frac{\nu}{2}\right)$. According to Eq. 33 in Ref. [2], the spectral density reads $S(f) \sim \frac{1}{f^\beta}$, $\beta = 1 + \frac{\nu-3}{2(m-1)}$. Inserting $\nu = 2(m-c)$, we obtain that for the process in Eq. (17) $\beta(m) = 2 - \frac{1+2c}{2(m-1)}$, which gives the result Eq. (16), putting $\nu(m) = \frac{1}{2(m-1)}$ and $\epsilon = 2c$. □

Note that for $m = \frac{3}{2}$ and $c = 0$, the power-spectrum shows pure $1/f$ behavior, see Fig. 8, while for $m = \frac{5}{4}$, the spectrum is flat, $\beta = 0$. The n-dependence of the spectral density shows up only for small f. One can show that $S(f)$ for small f is an increasing function of n.

3. Bursts generated in the gCEV process

Regarding the transformed gCEV process Y_t for $n < n_*$ as a diffusion being trapped in a convex potential $U(y)$ as in Fig. 5, makes it clear that the dynamics of X_t allows for a sequence of arbitrary high but finite outbursts even on short time scales, in agreement with Fig. 1. Since the bursting behavior, i.e. X_t large, is governed by the state-to-diffusion feedback parameter m, we can restrict ourselves to the case $n = 1$ for investigating statistical properties of burst. That is, we will numerically consider the CEV process

$$\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t^{\frac{3}{2}}\mathrm{d}W \tag{18}$$

in the following. Kaulakys and Alaburda [7] considered the case $m = 2$ in $\mathrm{d}X = \frac{a}{b^2}x^{2m-1}\mathrm{d}t + x^m\mathrm{d}W$, for $x \geq x_m > 0$. Note that is the case X is distributed according to a power law with tail exponent $2\left(m - \frac{a}{b^2}\right)$, as follows from Eq. (3). In this particular setting they found numerical evidences for clear power-law statistics of bursts. Since in our case, power-law behavior only exists asymptotically, we can expect power-law burst statistics only asymptotically.

A BURST is regarded as a super-threshold event: Let (X_t) be the solution of a gCEV process. The *burst interval* T_k of the k-th burst is defined as the time interval between crossing the threshold $\underline{x} > 0$ from below and the smallest time at which the threshold is crossed back from above. By a slight abuse of notation we also denote the length of this burst period by T_k. In Fig. 9 its probability distributions $P_{\underline{x}}(T) = P(T_k = T \mid \underline{x})$ are shown for different threshold values: red $\underline{x} = 2$, green $\underline{x} = 4$, blue $\underline{x} = 8$. The distribution $P(T)$ of burst durations admits an intermediary power-law regime with $\frac{1}{T^{\frac{3}{2}}}$.

The size of a burst is defined as $S = \int_T X_t\mathrm{d}t$, i.e. the integral over the super-threshold trajectory in the burst period T. Size turns out to be related to the burst duration T by

$$S \propto T^2 \tag{19}$$

as can be seen in the double-logarithmic plot in Fig. 10, where the common slope is 2.

4. Conclusion

We studied a generalization of the well-known Constant-Elasticity-of-Volatility (CEV) process; the *generalized CEV process* (gCEV) is given by $\mathrm{d}X_t = aX_t^n + bX_t^m\mathrm{d}W_t$, where $a, b > 0$. Dynamics following a gCEV process is due to the interplay

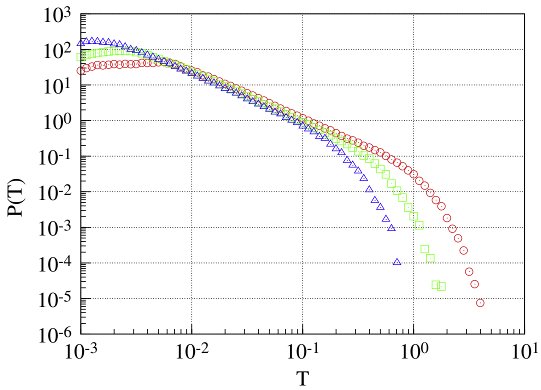

Fig. 9. Intermediary power-law regime in $P_{\underline{x}}(T)$ for different threshold values: $\underline{x} = 2$ (red circles), $\underline{x} = 4$ (green squares), $\underline{x} = 8$ (blue triangles).

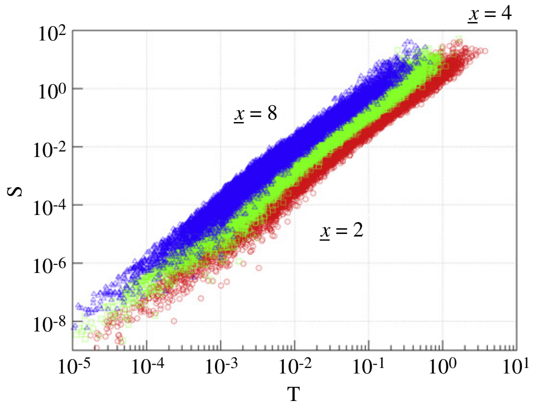

Fig. 10. Size S of a burst and its duration T for three thresholds, $\underline{x} = 2$ (red circles), $\underline{x} = 4$ (green squares), $\underline{x} = 8$ (blue triangles).

between two feedbacks: feedback in drift and feedback in diffusion. The emphasis was on the case that *both* feedback mechanisms are positive, i.e. $1 < n, m < \infty$. While, in general, positive feedback leads to singular behavior, we found that both positive feedbacks can play together so that system dynamics has a stationary probability density function $P(x)$. This stationary pdf in fact exists if both positive feedbacks are 'balanced' in the sense that given a positive drift feedback $n > 1$, the diffusion feedback must be sufficiently strong, i.e. $m > \frac{1}{2}(n+1)$. In this case $P(x)$ decays according to a power-law for large x according to $P(x) \sim \frac{1}{x^{2m}}$. For $m = 3/2$ one obtains that the stationary pdf obeys a cubic law.

$$P(x) \sim \frac{1}{x^3}$$

Consequently the power-spectral density of the gCEV process with $n, m > 1$ and $n < 2m - 1$ also decays as a power-law asymptotically with $S(f) \sim \frac{1}{f^{\beta(m)}}$, where $b(m) \leq 2$ for $m > 1$ and increasing, see Eq. (16). Particularly, for $m = \frac{3}{2}$ one obtains asymptotic pure $1/f$ noise. The tail behavior of the stationary pdf as well as of the power-spectral density are both independent of the drift feedback degree n but governed by the diffusion feedback degree m.

Finally we studied the 'outburst' behavior of gCEFV processes numerically. Since gCEV processes admit sequences of arbitrary high but finite excursions, our interest was on the duration T of bursts and their intensity S. We found a simple quadratic correspondence – in distribution –

$$S \propto T^2.$$

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