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Stochastic nonlinear differential equation generating 1/*f* noise

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Starting from the simple point process model of $1/f$ noise, we derive a stochastic nonlinear differential equation for the signal exhibiting $1/f$ noise, in any desirably wide range of frequency. A stochastic differential equation (the general Langevin equation with a multiplicative noise) that gives $1/f$ noise is derived. The solution of the equation exhibits the power-law distribution. The process with $1/f$ noise is demonstrated by the numerical solution of the derived equation with the appropriate restriction of the diffusion of the signal in some finite interval.

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The power spectra of a large variety of systems ranging widely from astrophysics and technology to sociology and psychology at low frequencies have $1/f$ behavior, i.e., the power density $S(f)$ is inversely proportional to the frequency f [1–9]. $1/f$ noise, also known as flicker noise, is intermediate between white noise [no correlation in time, $S(f) \sim 1/f^0$] and Brownian motion [no correlation between increments, $S(f) \sim 1/f^2$]. Simple procedures of integration or differentiation of such fluctuating signals do not yield the signal exhibiting $1/f$ noise. Most of the $1/f$ noise models are specialized or complicated. This makes the problem of omnipresent $1/f$ noise one of the oldest puzzles in contemporary physics. In contrast to the Brownian motion generated by the linear stochastic equation, simple systems of differential, even linear stochastic equations generating signals with $1/f$ noise are not known.

The purpose of this paper is the derivation of a nonlinear stochastic differential equation (generalized Langevin equation for the signal) generating a signal with $1/f$ noise. The stochastic differential equation is obtained from the point process model of $1/f$ noise, analyzed in Refs. [10–16]. Such a method enables one to obtain various stochastic differential equations, starting from different point processes and generating stochastic signals with different slopes of the power density. Analysis of the concrete physical models and application of the derived nonlinear stochastic equation for modeling of the specific observable processes are beyond the scope of this paper.

We start from the point process model recently proposed and analyzed in Refs. [10–16]. The signal in the model consists of pulses or series of events,

$$I(t) = a\sum_k \delta(t - t_k). \tag{1}$$

Here $\delta(t)$ is the Dirac delta function, $\{t_k\}$ is a set of the occurrence times at which the particles or pulses cross the section of observation, and a is a contribution to the signal of one pulse or particle. The power spectral density of the point process (1) may be expressed as [10–16]

$$S(f) = \lim_{T\to\infty}\left\langle \frac{2a^2}{T}\sum_{k,q} e^{i2\pi f\Delta(k;q)}\right\rangle, \tag{2}$$

where T is the observation time and

$$\Delta(k;q) \equiv t_{k+q} - t_k = \sum_{l=k}^{k+q-1} \tau_l \tag{3}$$

is the difference between the pulses occurrence times t_{k+q} and t_k. Here the brackets $\langle\cdots\rangle$ denote the averaging over the realizations of the process and $\tau_k = t_{k+1} - t_k$ is the interevent time. In the model [10–16], the interevent time of the signal stochastically diffuses about some average value and the process has been described by an autoregressive iteration with a very small relaxation. Here we will consider the stochastic point process described by the recurrent equations

$$t_{k+1} = t_k + \tau_k, \tag{4}$$

$$\tau_{k+1} = \tau_k + \sigma\varepsilon_k \tag{5}$$

with the appropriate boundary conditions, restricting the diffusion of τ_k in the finite interval $[\tau_{\min}, \tau_{\max}]$. In Eq. (5), ε_k are normally distributed uncorrelated random variables with a zero expectation and unit variance, i.e., a white noise, and σ is a standard deviation of the white noise.

The signal (1) generated according to Eqs. (4) and (5), depending on the parameter σ and the interval $[\tau_{\min}, \tau_{\max}]$, exhibits $1/f$ noise in any desirably wide range of frequency. According to the general theory [10–16], the power spectral density of such a point process for $f \lesssim \tau_{\max}^{-1}$ and $\tau_{\min} \to 0$ may be estimated as

$$S(f) \sim \frac{a^2}{\tau_{\max}^2}\frac{1}{f}. \tag{6}$$

The spectrum obtained from the numerical solution of Eqs. (4) and (5) with reflective boundary conditions at $\tau_{\min}$ and $\tau_{\max}$ is shown in Fig. 1. We see that the considered point process gives $1/f$ noise in a wide range of frequencies.

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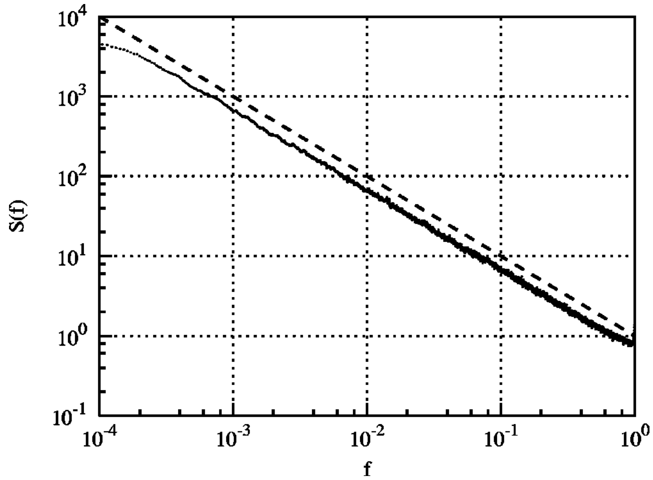

FIG. 1. Power spectral density of the point process, described by Eqs. (4) and (5). Parameters used are $\sigma=0.01$, $\tau_{\min}=10^{-5}$, and $\tau_{\max}=1$. The dashed line represents the spectral density, calculated according to the equation $S(f)=1/f$.

It is the purpose of this paper to derive a stochastic differential equation for the signal, the solution of which exhibits $1/f$ noise. For this purpose, we rewrite Eq. (5) as a differential Ito stochastic equation interpreting k as a continuous variable, i.e.,

$$\frac{d\tau_k}{dk} = \sigma\xi(k). \tag{7}$$

Here $\xi(k)$ is a Gaussian white noise satisfying the standard condition

$$\langle \xi(k)\xi(k')\rangle = \delta(k-k'). \tag{8}$$

Then we rewrite Eq. (7) using the occurrence time. Transition from the occurrence number k to the actual time t according to the relation $dt=\tau_k dk$ yields the equation

$$\frac{d\tau}{dt} = \frac{\sigma}{\sqrt{\tau}}\xi(t). \tag{9}$$

The signal averaged over the time interval τ_k according to Eq. (1) is $x=a/\tau_k$. The standard [17] transformation of the variable from τ to $x=a/\tau$ in Eq. (9) results in the stochastic differential Ito equation

$$\frac{dx}{dt} = \frac{\sigma^2}{a^3}x^4 + \frac{\sigma}{a^{3/2}}x^{5/2}\xi(t). \tag{10}$$

Equation (10) can be rewritten in a form that does not contain any parameters. Introducing the scaled time

$$t_s = \frac{\sigma^2}{a^3}t, \tag{11}$$

we obtain from Eq. (10) an equation

$$\frac{dx}{dt_s} = x^4 + x^{5/2}\xi(t_s). \tag{12}$$

The steady-state solution of the stationary Fokker-Planck equation with the appropriate reflective boundary conditions and a zero flow obtained from Eq. (12) according to the standard method [17] is of the power-law form,

$$P(x) = \frac{C}{x^3}, \tag{13}$$

where C has to be defined from the normalization.

The power-law distribution of the signals is the phenomenon observable in a large variety of processes, from earthquakes to the financial time series [7,8,18,19]. Therefore, our model of $1/f$ noise is complementary to the models based on the superposition of signals with a wide-range distribution of the relaxation times resulting in the Gaussian process [20].

Because of the divergence of the power-law distribution and the requirement of the stationarity of the process, the stochastic equation (12) should be analyzed together with the appropriate restrictions of the diffusion in some finite interval $x_{\min}\lessapprox x\lessapprox x_{\max}$. Such restrictions may be introduced as some additional conditions to the iterative solution of the stochastic differential equation. Similar restrictions, however, may be fulfilled by introducing some additional terms into Eq. (12), corresponding to the restriction of the diffusion in some "potential well." According to the general theory [17], the exponentially restricted diffusion with the distribution density

$$P(x) \sim \frac{1}{x^3}\exp\left\{-\left(\frac{x_{\min}}{x}\right)^n - \left(\frac{x}{x_{\max}}\right)^n\right\} \tag{14}$$

generates the stochastic differential equation

$$\frac{dx}{dt_s} = \frac{n}{2}\left(\frac{x_{\min}^n}{x^{n-4}} - \frac{x^{n+4}}{x_{\max}^n}\right) + x^4 + x^{5/2}\xi(t_s). \tag{15}$$

Here n is some parameter.

Equations (12) and (15) are the main result of this paper. Since the point process (4) and (5) gives the signal with $1/f$ noise, the signal obtained from Eqs. (12) and (15) also should give $1/f$ noise in some frequency interval. When $x_{\max}\to\infty$, from Eq. (6) we can estimate the power spectral density as

$$S(f) \sim x_{\min}^2\frac{1}{f}. \tag{16}$$

Such a conclusion is confirmed by the numerical solution of Eq. (15).

We solve Eqs. (12) and (15) using the method of discretization. When the variable step of integration is $\Delta t_s=h_i$, the differential equation (15) transforms to the difference equation

$$x_{i+1} = x_i + \frac{n}{2}\left(\frac{x_{\min}^n}{x_i^{n-4}} - \frac{x_i^{n+4}}{x_{\max}^n}\right)h_i + x_i^4 h_i + x_i^{5/2}\sqrt{h_i}\varepsilon_i. \tag{17}$$

We can solve Eq. (17) numerically with the constant step, $h_i=\text{const}$, when $t_{i+1}=t_i+h$. However, one of the most effective methods of solutions of Eq. (17) is when the change of the variable x_i in one step is proportional to the value of the variable. We take the integration steps h_i from the equation $x_i^{5/2}\sqrt{h_i}=\kappa x_i$, with $\kappa\ll 1$ being a small parameter. As a result, we have the system of equations

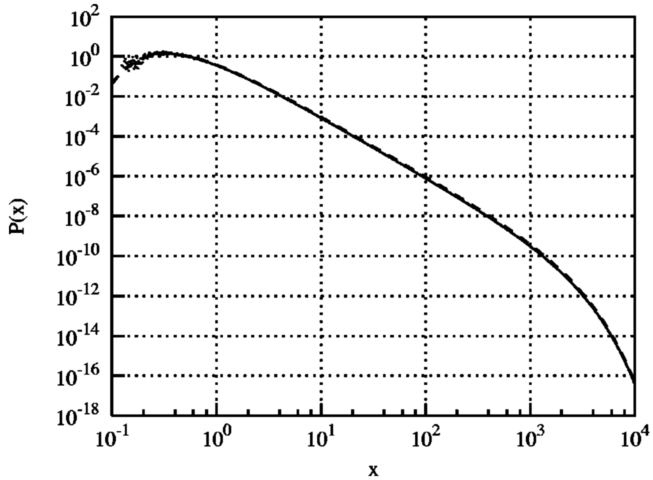

FIG. 2. Distribution density $P(x)$ obtained from the numerical solution of Eq. (18). The dashed line represents the distribution density calculated from Eq. (14). Parameters used are $x_{\min}=1$, $x_{\max}=10^3$, $n=1$, and $\kappa=0.1$.

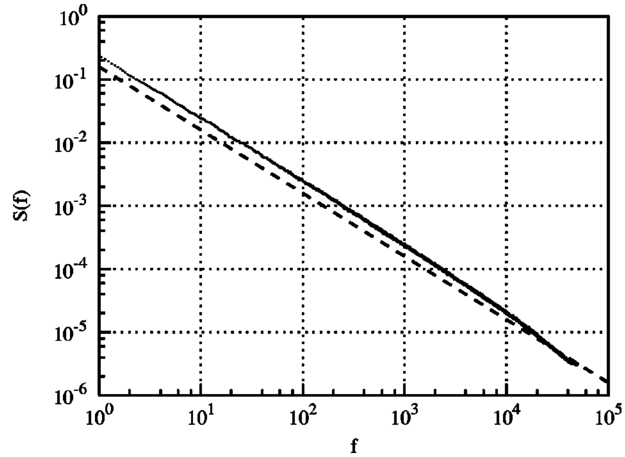

FIG. 3. Power spectral density of the signal, obtained from Eq. (18). The dashed line represents the spectral density, calculated according to equation $S(f)=1/(2\pi f)$. The parameters used are the same as in Fig. 2.

$$x_{i+1} = x_i + \kappa^2 x_i \left[1 + \frac{n}{2} \left(\frac{x_{\min}^n}{x_i^n} - \frac{x_i^n}{x_{\max}^n} \right) \right] + \kappa x_i \varepsilon_i,$$

$$t_{i+1} = t_i + \frac{\kappa^2}{x_i^3}. \tag{18}$$

The distribution density $P(x)$ of the variable x, obtained using Eq. (18), is shown in Fig. 2. We see that our method of solution gives good agreement with the power-law distribution (13) in the interval $x_{\min} \lesssim x \lesssim x_{\max}$.

The power spectral density $S(f)$ calculated according to Eqs. (18) is shown in Fig. 3. Figure 3 shows that Eq. (15) indeed gives a signal exhibiting $1/f$ noise in a wide frequency interval.

In summary, we derived a stochastic differential equation for the signal exhibiting $1/f$ noise in any desirably wide range of frequency. The distribution density of the signal is of the inverse cubic power law. The numerical analysis of the obtained equation shows that the signal indeed exhibits $1/f$ noise and power-law distribution.

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