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NONEXTENSIVE STATISTICAL MECHANICS DISTRIBUTIONS AND DYNAMICS OF FINANCIAL OBSERVABLES FROM THE NONLINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

JULIUS RUSECKAS*, VYGINTAS GONTIS
and BRONISLOVAS KAULAKYS

Institute of Theoretical Physics and Astronomy,
Vilnius University,
A. Goštauto 12, LT-01108 Vilnius, Lithuania
**julius.ruseckas@tfai.vu.lt*

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We present nonlinear stochastic differential equations, generating processes with the q-exponential and q-Gaussian distributions of the observables, i.e. with the long-range power-law autocorrelations and $1/f^{\beta}$ power spectral density. Similarly, the Tsallis q-distributions may be obtained in the superstatistical framework as a superposition of different local dynamics at different time intervals. In such approach, the average of the stochastic variable is generated by the nonlinear stochastic process, while the local distribution of the signal is exponential or Gaussian one, conditioned by the slow average. Further we analyze relevance of the generalized and adapted equations for modeling the financial processes. We model the inter-trade durations, the trading activity and the normalized return using the superstatistical approaches with the exponential and normal distributions of the local signals driven by the nonlinear stochastic process.

Keywords: Stochastic differential equations; Tsallis distributions; superstatistics; financial systems.

1. Introduction

Time series of financial data exhibit nontrivial statistical properties. Many of these anomalous properties appear to be universal and a variety of the so-called stylized facts has been established [43, 11, 45, 12, 44, 26, 15, 7, 66]. The probability density function (PDF) of the return and other financial variables are successfully described by the distributions of the nonextensive statistical mechanics [58, 17, 9, 21, 23]. The return has a distribution that is very well fitted by q-Gaussians, only slowly becoming Gaussian as the time scale approaches months, years and longer time scale. Another interesting statistics which can be modeled within the nonextensive framework is the distribution of volumes, defined as the number of shares traded.

Many complex systems show large fluctuations that follow non-Gaussian, heavy-tailed distributions with the power-law temporal correlations, scaling and the fractal features [42, 44, 41]. These distributions, scaling, self-similarity and fractality are often related with the Tsallis nonextensive statistical mechanics [60, 48, 61, 62, 57] and with $1/f^{\beta}$ noise (see, e.g. [41, 30, 29, 14, 37, 54], and references herein). Often nonextensive statistical mechanics represents a consistent theoretical background for the investigation of complex systems, [61, 62, 57]. On the other hand, a usual way to describe stochastic evolution and the properties of complex systems is by the stochastic differential equations (SDE) [16, 52, 13, 64, 65]. Such nondeterministic equations of motion are used for modeling the financial systems, as well [38, 44, 27, 20, 19, 21, 22].

There are empirically established facts that the trading activity, trading volume, and volatility are stochastic variables with the long-range correlation [10, 46, 15]. However, these aspects are not accounted for in some widely used models of the financial systems. Moreover, the trading volume and the trading activity are positively correlated with the market volatility, while the trading volume and volatility show the same type of the long memory behavior [40], including $1/f^{\beta}$ noise [21, 22].

The purpose of this article is to model the inter-trade durations, the trading activity and the normalized return using the superstatistical approaches with the exponential and normal distributions of the local signals driven by the nonlinear stochastic process. We present a class of nonlinear stochastic differential equations giving the power-law behavior of the probability density function (PDF) of the signal intensity and of the power spectral density ($1/f^{\beta}$ noise) in any desirably wide range of frequency. Modifications of these equations by introducing an additional parameter yields Tsallis distributions preserving $1/f^{\beta}$ behavior of the power spectral density. The superstatistical framework [4, 59, 1, 2, 63, 24, 5] using a fast dynamics with the slowly changing parameter described by nonlinear stochastic differential equations can yield q-exponential or q-Gaussian long-term stationary PDF of the signal retaining the long-range correlations, as well.

Further, we apply these approaches for modeling the dynamics of the financial observables, as far as they are long-range depending and usually may be well approximated by q-distributions. In contrast to the widely used models of the financial market dynamics, the present model account for the long-range correlations of the observables, exhibiting as well in the $1/f^{\beta}$ spectra. The finding that the market volatility is proportional to the trading activity allows us to describe both appearances using the same parameters of the model, based on the nonlinear SDEs and superstatistical framework.

2. Modeling Power-Law Processes by the Nonlinear Stochastic Differential Equations

The nonlinear SDEs generating power-law distributed processes with $1/f^{\beta}$ noise have been derived in papers [33, 32, 29] starting from the point process model

[31, 34, 35, 30]. The general expression of Itô SDEs is

$$dx = \sigma^2 \left(\eta - \frac{1}{2}\lambda\right) x^{2\eta-1} dt + \sigma x^{\eta} dW. \tag{1}$$

Here x is the signal, $\eta \neq 1$ is the exponent of the power-law multiplicative noise, λ defines the exponent of the steady-state PDF of the signal in some interval, $P(x) \sim x^{-\lambda}$, and W is a standard Wiener process. Some motivations of Eq. (1) have been given in [33, 30, 32, 29, 28, 54]. In papers [53, 36] the nonlinear SDE of type (1) has been obtained starting from a simple agent-based model describing the herding behavior.

In order to obtain a stationary process and avoid the divergence of steady-state PDF the diffusion of stochastic variable x should be restricted at least from the side of small values, or Eq. (1) should be modified. Modifications of these equations by introducing of an additional parameter x_0 are presented in [54]. The PDF of the signal generated by modified SDEs

$$dx = \sigma^2 \left(\eta - \frac{1}{2}\lambda\right) (x + x_0)^{2\eta-1} dt + \sigma (x + x_0)^{\eta} dW \tag{2}$$

and

$$dx = \sigma^2 \left(\eta - \frac{1}{2}\lambda\right) (x^2 + x_0^2)^{\eta-1} x dt + \sigma (x^2 + x_0^2)^{\eta/2} dW \tag{3}$$

is q-exponential and q-Gaussian distribution of the nonextensive statistical mechanics, respectively. Stochastic differential Eqs. (2) and (3) for small $x \ll x_0$ represent the linear additive stochastic process generating the Brownian motion with the steady drift or linear relaxation, respectively, avoiding the power-law divergence of the signal distribution when $x \to 0$, while for $x \gg x_0$ they reduce to the multiplicative SDE (1) and preserve $1/f^{\beta}$ behavior of the power spectral density. In [32, 29, 54] it was shown that SDEs (1)–(3) generate signals with the power spectral density

$$S(f) \sim \frac{1}{f^{\beta}}, \quad \beta = 1 + \frac{\lambda - 3}{2(\eta - 1)} \tag{4}$$

in a wide interval of frequencies.

3. Superstatistical Framework

Many nonequilibrium systems exhibit fluctuations of some parameters. The actual value of the signal x may be random variable with the distribution depending on the parameters and, consequently, on the slowly varying average $\bar{x}$. In such superstatistical approach [4, 59, 1, 24, 5, 64, 55, 65] the distribution $P(x)$ of the signal x is a superposition of the conditional distribution $\varphi(x|\bar{x})$ and the local stationary distribution $p(\bar{x})$ of the parameter $\bar{x}$,

$$P(x) = \int_0^\infty \varphi(x|\bar{x}) p(\bar{x}) d\bar{x} \,. \tag{5}$$

The superstatistical framework has successfully been applied on a widespread of problems, including economics [3, 49, 51, 50, 8, 2, 63].

We consider the model when the fluctuating parameter $\bar{x}$ evolves according to SDE (1) with the exponential restriction of diffusion. Using the superstatistical approach, it is possible in such a case to obtain the Tsallis probability distributions.

In order to obtain q-exponential PDF of the signal x we consider exponential PDF conditioned to the local average value of the parameter $\bar{x}$,

$$\varphi(x|\bar{x}) = \bar{x}^{-1} \exp(-x/\bar{x}). \tag{6}$$

A Poissonian-like process with slowly diffusing time-dependent average interevent time was considered in [28]. The mean $\bar{x}$ of the distribution $\varphi(x|\bar{x})$ obeys SDE with exponential restriction of diffusion,

$$d\bar{x} = \sigma^2 \left[\eta - \frac{\lambda}{2} + \frac{1}{2}\frac{x_0}{\bar{x}}\right] \bar{x}^{2\eta-1} dt + \sigma \bar{x}^{\eta} dW. \tag{7}$$

Here x_0 is a parameter describing exponential cut-off of the steady-state PDF of $\bar{x}$ at small values of $\bar{x}$. The steady-state PDF obtained from the Fokker–Planck equation corresponding to Eq. (7) is

$$p(\bar{x}) = \frac{1}{x_0 \Gamma(\lambda - 1)} \left(\frac{x_0}{\bar{x}}\right)^{\lambda} \exp\!\left(-\frac{x_0}{\bar{x}}\right). \tag{8}$$

Using Eqs. (5), (6) and (8), we get that the long-term stationary PDF of the signal x is q-exponential function,

$$P(x) = \frac{\lambda - 1}{x_0} \left(\frac{x_0}{x + x_0}\right)^{\lambda} = \frac{\lambda - 1}{x_0} \exp_q(-\lambda x/x_0), \quad q = 1 + 1/\lambda. \tag{9}$$

Here $\exp_q(\cdot)$ is the q-exponential function defined as

$$\exp_q(x) \equiv [(1 + (1 - q)x)]_+^{\frac{1}{1-q}}, \tag{10}$$

with $[(\ldots)]_+ = (\ldots)$ if $(\ldots) > 0$, and zero otherwise. Asymptotically, as $x \to \infty$, $\exp_q(x) \sim x^{-\lambda}$.

In order to obtain the q-Gaussian PDF of the signal x we consider the local Gaussian stationary PDF of x conditioned of the parameter $\bar{x}$ in Gaussian form,

$$\varphi(x|\bar{x}) = \frac{1}{\sqrt{\pi}\bar{x}} \exp(-x^2/\bar{x}^2). \tag{11}$$

The fluctuating parameter $\bar{x}$ obeys SDE

$$d\bar{x} = \sigma^2 \left[\eta - \frac{\lambda}{2} + \frac{x_0^2}{\bar{x}^2}\right] \bar{x}^{2\eta-1} dt + \sigma \bar{x}^\eta dW \tag{12}$$

with exponential restriction of diffusion. The steady-state PDF from the Fokker–Planck equation corresponding to Eq. (12) is

$$p(\bar{x}) = \frac{1}{x_0 \Gamma(\frac{\lambda-1}{2})} \left(\frac{x_0}{\bar{x}}\right)^\lambda \exp\left(-\frac{x_0^2}{\bar{x}^2}\right). \tag{13}$$

From Eqs. (5), (11) and (13) we obtain that the long-term stationary PDF of the signal x is q-Gaussian, i.e.

$$\begin{aligned} P(x) &= \frac{\Gamma(\frac{\lambda}{2})}{\sqrt{\pi} x_0 \Gamma(\frac{\lambda-1}{2})} \left(\frac{x_0^2}{x_0^2 + x^2}\right)^{\frac{\lambda}{2}} \\ &= \frac{\Gamma(\frac{\lambda}{2})}{\sqrt{\pi} x_0 \Gamma(\frac{\lambda-1}{2})} \exp_q\left(-\lambda \frac{x^2}{2x_0^2}\right), \quad q = 1 + 2/\lambda. \end{aligned} \tag{14}$$

4. Application for Trading Modeling

Now we analyze the tick by tick trades of 24 stocks, ABT, ADM, BMY, C, CVX, DOW, FNM, GE, GM, HD, IBM, JNJ, JPM, KO, LLY, MMM, MO, MOT, MRK, SLE, PFE, T, WMT, and XOM, traded on the NYSE for 27 months from January, 2005, and recorded in the Trades and Quotes database. PDF of intertrade durations τ for large values of τ are close to the stretched exponential distribution. In Fig. 1(a) we show the PDF of the intertrade durations for ABT stock. For other stocks this PDF exhibits similar shape. In order to show the stretching exponent more clearly we also plotted in Fig. 1(b) the logarithm of the PDF, as well. The obtained

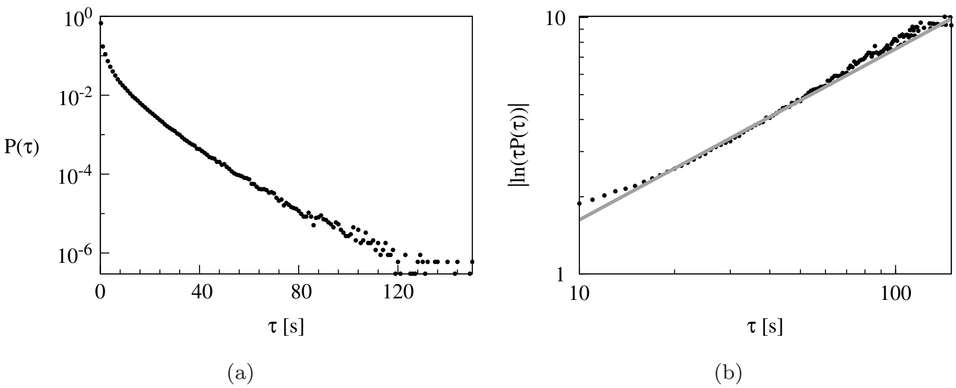

Fig. 1. PDF of intertrade durations for large values of duration τ for ABT stock (a) and the logarithm of the PDF weighted by intertrade duration τ (b). The slope of the gray line is $2/3 \approx 0.67$.

empirical values of the stretching exponent are similar for all stocks and close to 0.7. Note that PDF $P(\tau)$ of the intertrade durations weighted by τ, as used in Fig. 1, gives the distribution of intertrade durations "in real time" [30, 19], i.e. if we consider intertrade durations as a continuous function of time t then this function retains the value equal to τ during the time interval of the length τ.

We model the trading activity using superstatistical approach. Each individual intertrade duration τ is distributed according to exponential, Poissonian-like, distribution [20, 19]

$$\varphi(\tau|n) = n\exp(-n\tau). \tag{15}$$

Here, $n = 1/\bar{\tau}$ being the inverse of the average intertrade duration $\bar{\tau}$ is the slowly changing in time parameter. The meaning of the parameter n is the trading activity defined as number of trades in unit time. The model should be self-consistent: the trading activity $n(t)$ depends on time t, whereas time t is equal to the sum of intertrade durations, $t = \sum_k \tau_k$. We assume that PDF of n has power-law form for large n and has exponential cut-off for small n. Therefore, we choose PDF of n as

$$P_n(n) = \frac{2}{n_0\Gamma(\frac{\lambda-1}{2})}\left(\frac{n_0}{n}\right)^{\lambda}\exp\left(-\frac{n_0^2}{n^2}\right). \tag{16}$$

PDF of this form can be obtained from our SDE (12) with exponential restrictions at minimal, $n = n_0$, and maximal, $n = n_{\mathrm{max}}$, values of the trading activity

$$dn = \sigma^2\left[\eta - \frac{\lambda}{2} + \frac{n_0^2}{n^2} - \frac{n^2}{n_{\mathrm{max}}^2}\right]n^{2\eta-1}dt + \sigma n^{\eta}dW. \tag{17}$$

The parameter $n_{\mathrm{max}} \gg n_0$ leads to the exponential cut-off at very large values of the trading activity n. When $n \ll n_{\mathrm{max}}$ the influence of this cut-off is vanishing small.

The long-term PDF of intertrade durations is

$$P(\tau) = \int_0^\infty \frac{n}{\langle n\rangle}\varphi(\tau|n)P_n(n)dn. \tag{18}$$

Here

$$\langle n\rangle = \int_0^\infty P_n(n)ndn \tag{19}$$

is the average trading activity. Note, that in Eq. (18) the distribution of individual intertrade duration $\varphi(\tau|n)$ should be weighted by a number of trades, proportional to n. At large $\tau \gg \tau_0 = 1/n_0$, from Eqs. (15), (16) and (18) we have the asymptotic form of the integral (18)

$$P(\tau) \underset{\tau\to\infty}{\longrightarrow} \left(\frac{\tau}{\tau_0}\right)^{\frac{\lambda-4}{3}}\exp\left[-3\left(\frac{\tau}{2\tau_0}\right)^{\frac{2}{3}}\right]. \tag{20}$$

Therefore, we obtain that Eq. (18) automatically yields the stretching exponent equal to 2/3, which is close to the empirically determined values (see Fig. 1). It should be noted, that the stretching exponent is a sufficiently good approximation of the Weibull distribution, widely used distribution for modeling extreme events in long-term memory processes [6, 18, 47]. On the other hand, the power laws can be approximated by the Weibull distribution in arbitrary intervals to any prescribed accuracy [56, 6].

Comparison of the proposed PDF of the intertrade duration with empirical PDF for ABT stock is shown in Fig. 2. Using the appropriate parameters λ and τ_0 we obtain good agreement for all stocks.

The return $\tilde{r}$ over a time interval τ_d is defined as

$$\tilde{r}(t, \tau_d) = \ln p(t + \tau_d) - \ln p(t), \tag{21}$$

where $p(t)$ is the stock price. Instead of unnormalized return $\tilde{r}$ we will use the normalized return

$$r = \frac{\tilde{r} - \langle \tilde{r} \rangle}{\sqrt{\langle (\tilde{r} - \langle \tilde{r} \rangle)^2 \rangle}}. \tag{22}$$

The PDF of the normalized return per 1 min is close to q-Gaussian, as is shown in Fig. 3. In order to account for the double stochastic nature of the return fluctuations — a hidden slowly diffusing long-range memory process and rapid fluctuations of the instantaneous price changes [20, 19, 21] — we use a superstatistical approach for the return. Detailed analysis of the empirical data from the NYSE provides evidence that the long-range memory properties of the return strongly depend on the fluctuations of the trading activity. Therefore, in our model we will assume that standard deviation of the local PDF of return is proportional to n. We will take a normal distribution with mean equal to zero and variance $(an)^2$ as the local PDF

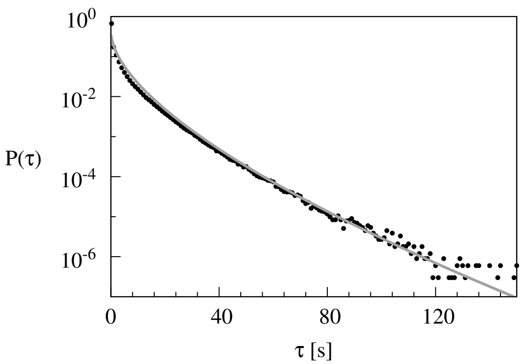

Fig. 2. PDF of intertrade duration for large values of duration τ for ABT stock. Gray line is calculated according to Eqs. (15), (16) and (18). Used parameters are $\lambda = 3.4$, $\tau_0 = 7.3\,\mathrm{s}$.

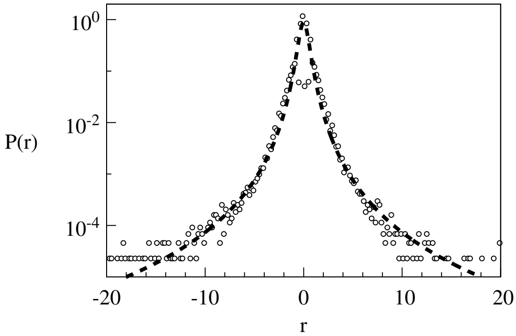

Fig. 3. PDF of normalized return per 1 min for ABT stock. The dashed line is q-Gaussian distribution according to Eq. (25) with parameters $\lambda = 3.4$ ($q = 1.59$).

of return conditioned to value of the parameter n,

$$\varphi(r|n) = \frac{1}{\sqrt{\pi}an} \exp\left(-\frac{r^2}{a^2n^2}\right). \tag{23}$$

Here a is a coefficient of proportionality.

The long-term PDF of return is given by the equation

$$P(r) = \int_0^\infty \varphi(r|n)P_n(n)dn. \tag{24}$$

Using the same distribution (16) as in modeling PDF of the intertrade duration, we get PDF of return

$$\begin{aligned} P(r) &= \frac{\Gamma(\frac{\lambda}{2})}{\sqrt{\pi}r_0\Gamma(\frac{\lambda-1}{2})}\left(\frac{r_0^2}{r_0^2+r^2}\right)^{\frac{\lambda}{2}} \\ &= \frac{\Gamma(\frac{\lambda}{2})}{\sqrt{\pi}r_0\Gamma(\frac{\lambda-1}{2})}\exp_q\left(-\lambda\frac{r^2}{2r_0^2}\right), \quad q = 1+\frac{2}{\lambda}. \end{aligned} \tag{25}$$

Here $r_0 = an_0$. Since the return r defined according to Eq. (22) is normalized to the variance equal 1, the variance and the standard deviation of r in Eq. (25) should be equal to 1, as well. From this condition it follows

$$r_0 = \sqrt{\lambda - 3}. \tag{26}$$

Equation (25) shows that our model gives q-Gaussian long-term PDF of return. Comparison of the empirical distribution of normalized return per 1 min for ABT stock with the q-Gaussian distribution is shown in Fig. 3. We obtain a good agreement with the empirical PDF. We used the same parameter λ for both PDFs of the intertrade durations and of the trading activity. Thus PDF of intertrade duration and PDF of return can be obtained using superstatistical approach from trading activity, whereas trading activity may be modeled by proposed nonlinear SDE. Such approach readily gives power-law spectra of the sequence of trades and of the absolute return.

Now, as an example, we will consider SDE (17) with the exponent $\eta = 5/2$ as the equation for trading activity n. From the empirical data for ABT stock we get the parameters of the equation $\lambda = 3.4$ and $\tau_0 = 1/n_0 = 7.3\,\mathrm{s}$. From Eq. (4) it follows that the chosen parameters yield the exponent of the power spectral density $\beta = 1.13$. Comparison of the empirically obtained PDFs of intertrade duration and of normalized one-minute return with the ones from superstatistical model as well as comparison of empirical power spectral density of the sequence of trades and of absolute return with respective spectral densities from superstatistical model are is shown in Fig. 4. We see that this simple approach gives good agreement of distributions, whereas there is bigger discrepancy between spectra. Although our model reproduces the power-law behavior of the spectrum, the exponent of the power spectral density β is larger than 1, whereas the empirical data yield $\beta < 1$. More complicated equation investigated in [19, 21] corresponds to the empirical power spectral density of the sequence of trades more properly.

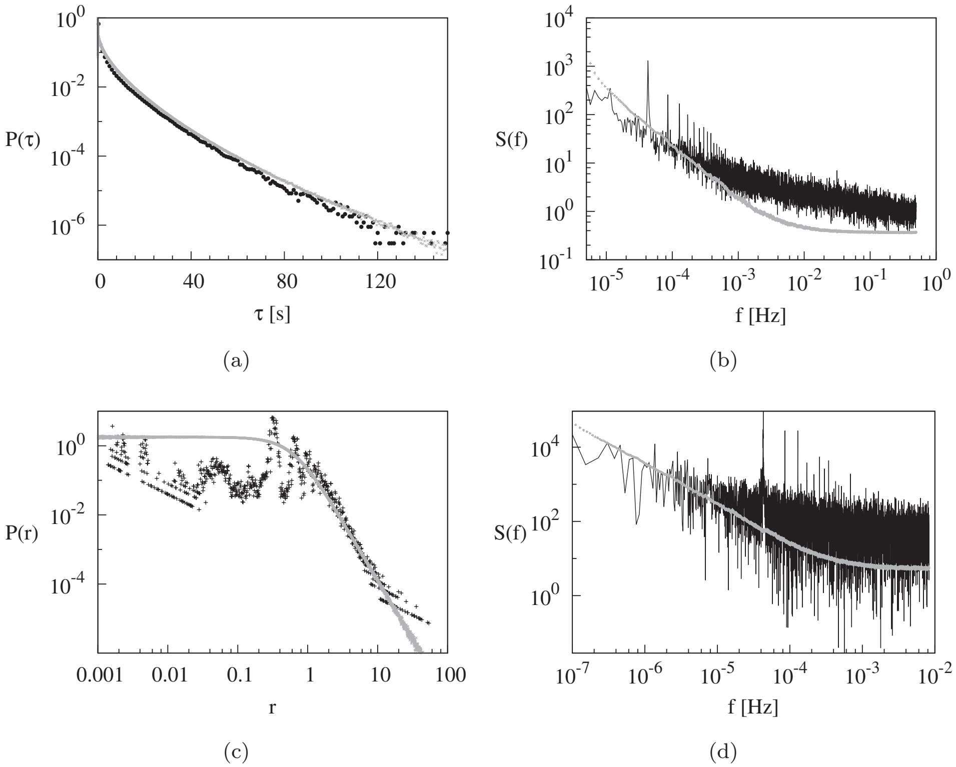

Fig. 4. PDF of intertrade duration (a), power spectral density of a sequence of trades (b), PDF (c) and power spectral density (d) of the absolute value of the normalized one-minute return for ABT stock. Both empirical and obtained from superstatistical model with SDE (17) and $\eta = 5/2$ (smooth gray curves) PDFs and spectra are shown. The parameters used for the model are $\tau_0 = 7.3\,\mathrm{s}$, $\lambda = 3.4$, $\sigma/\tau_0 = 0.001$, $n_{\max}/n_0 = 10^3$.

5. Conclusion

The complete description of a macroscopic system would consist of solving all the microscopic deterministic equations of the system. As it generally cannot be done, we use a stochastic description instead, i.e. we describe the system by macroscopic variables which fluctuate in a stochastic way [52, 16]. In nature, almost as a rule, most systems behave as the open ones, interacting with an environment. The evolution of these systems must then be nonunitary, i.e. interactions with the environment must lead to dissipation as well as in the stochastic effects, which is the way the environment back-reacts on the system. One common way for describing the above-mentioned forms of evolution is by means of generalized stochastic differential equations of motion [39, 13, 25].

In this paper we considered a class of nonlinear stochastic differential equations, giving asymptotically the power-law behavior of the distribution and of the power spectral density ($1/f^{\beta}$ noise). Modifications of the equations by introducing the additional parameter, which make the Brownian-like motion of the small variable and preserve from the divergence the power-law distribution at small signals, give q-probability distribution functions from the nonextensive statistical mechanics, preserving the exhibition of $1/f^{\beta}$ noise, as well. The superstatistical concept, consisting of the superposition of exponential or Gaussian distributions with the slow stochastically behaving and power-law distributed means of the process, yields the q-exponential and q-Gaussian distributions, respectively, with $1/f^{\beta}$ noise of the generated signal, as well.

Further we used such framework for modeling the financial systems. We showed that the inter-trade durations, the trading activity and the normalized return may be replicated using the superstatistical approaches with the exponential and normal distributions of the local signals driven by the nonlinear stochastic process of the means, i.e. as a nonlinear double stochastic process.

Summarizing, we have demonstrated that starting from the nonlinear stochastic differential equations it is possible to model the long-range processes with the power-law distributions, including q-distributions, which preserve the power-law divergence of the signal distribution when $x \to 0$. Combination of the superstatistical framework with the stochastic processes generated using the nonlinear SDEs yields the long-range nonlinear double stochastic processes with the rapid stochastic local variable and slow stochastic mean. Such an approach may be used for simulation of the dynamics of some financial observables. Analysis of the basis and universality of the nonextensive statistical mechanics itself is beyond the scope of this paper.

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