

Solutions of nonlinear stochastic differential equations with $1/f$ noise power spectrum

Bronislovas Kaulakys, Julius Ruseckas

Institute of Theoretical Physics and Astronomy, Vilnius University, A. Goštauto 12, LT-01108 Vilnius, Lithuania

Abstract—The special nonlinear stochastic differential equations generating power-law distributed signals and $1/f$ noise are considered. The models involve the generalized *Constant Elasticity of Variance* (CEV) process, the *Bessel process*, the *Squared Bessel process*, and the *Cox-Ingersoll-Ross* (CIR) process, which are applied for modeling the financial markets, as well. In the paper, $1/f^\beta$ behavior of the power spectral density is derived directly from the nonlinear stochastic differential equations and the exact solutions for the particular CEV process are presented.

I. INTRODUCTION

The phrases “ $1/f$ noise”, “ $1/f$ fluctuations”, and “flicker noise” refer to the phenomenon, having the power spectral density at low frequencies f of signals of the form $S(f) \sim f^{-\beta}$, with β being a system-dependent parameter. $1/f^\beta$ signals with $0.5 < \beta < 1.5$ are found widely in nature, occurring in physics, electronics, astrophysics, geophysics, economics, biology, psychology, language and even music [1]–[4] (see also references in recent paper [5]). The case of $\beta = 1$, or “pink noise”, is the one of the most interesting. The widespread occurrence of processes exhibiting $1/f$ noise suggests that a generic, at least mathematical explanation of such phenomena might exist.

Here we model $1/f$ noise using the nonlinear stochastic differential equations (SDEs). Nonlinear SDE with the linear noise and nonlinear drift was considered in Ref. [6]. Recently the nonlinear SDE, generating signals with $1/f$ noise was obtained in Refs. [7], [8] (see also recent paper [5]), starting from the point process model of $1/f$ noise [9]–[16]. In this paper, $1/f^\beta$ behavior of the power spectral density is derived from the nonlinear SDE, using the eigenfunction expansion of the associated Fokker-Planck equations. The exact solutions for the special *Constant Elasticity of Variance* (CEV) process are presented, as well.

II. STOCHASTIC DIFFERENTIAL EQUATION GENERATING SIGNAL WITH $f^{-\beta}$ NOISE

The fluctuations of the intensity of the signals, currents, flows, etc, consisting of discrete objects (electrons, photons, packets, vehicles, pulses, events, etc), are primarily and basically defined in terms of fluctuations of the (average) interevent, interpulse, interarrival, recurrence, or waiting time. For the process consisting of the discrete objects the intensity of the signal fluctuates due to the fluctuations of rate, i.e., density of the objects in the time axis, which is a consequence of fluctuations of the interarrival or interevent time.

We start from the point process $x(t) = a \sum_k \delta(t - t_k)$, representing the signal, current or flow, $x(t)$, as a sequence

of correlated pulses or series of events. Here $\delta(t)$ is the Dirac δ -function and a is an average contribution to the signal $x(t)$ of one pulse at the time moment t_k . Our model is based on the generic multiplicative process for the interevent time $\tau_k \equiv t_{k+1} - t_k$,

$$\tau_{k+1} = \tau_k + \gamma \tau_k^{2\mu-1} + \sigma_\tau \tau_k^\mu \varepsilon_k, \quad (1)$$

generating the power-law distributed $P_k(\tau_k) \sim \tau_k^\alpha$, with $\alpha = 2\gamma/\sigma_\tau^2 - 2\mu$, sequence of the interevent times τ_k and $f^{-\beta}$, with $\beta = 1 + \alpha/(3 - 2\mu)$, power spectral density of the signal [11]–[16]. Some motivations for equation (1) were given in papers [5], [9]–[19].

Therefore, in our model the (average) interevent time τ_k fluctuates due to the random perturbations by a sequence of uncorrelated normally distributed random variables $\{\varepsilon_k\}$ with the zero expectation and unit variance; σ_τ is standard deviation of the white noise and $\gamma \ll 1$ is a coefficient of the nonlinear damping. Transition from the occurrence number k to the actual time t in (1) according to the relation $dt = \tau_k dk$ yields the Itô SDE for the variable $\tau(t)$ as a function of the actual time,

$$d\tau = \gamma \tau^{2\mu-2} dt + \sigma_\tau \tau^{\mu-1/2} dW, \quad (2)$$

where W is a standard Wiener process. Equation (2) generates the stochastic variable τ , power-law distributed, $P_t(\tau) \sim \tau^{\alpha+1}$, in the actual time t . The power-law distribution of the interevent, recurrence, or waiting time is observed in different systems from physics and seismology to the Internet and financial markets (see, e.g., [11]–[16], [20]–[23]).

The Itô transformation in (2) of the variable from τ to the intensity (averaged over the time interval τ) of the signal $x(t) = a/\tau(t)$ [7], [8] yields the class of Itô SDE generating signals with $f^{-\beta}$ power spectral density:

$$dx = \sigma^2(\eta - \nu/2)x^{2\eta-1}dt + \sigma x^\eta dW. \quad (3)$$

Here the new parameters $\eta = 5/2 - \mu$, $\nu = 3 + \alpha$ and $\sigma = \sigma_\tau/a^{3/2-\mu}$ have been introduced. The exponent β of the power spectral density $S(f) \sim f^{-\beta}$ in the new parameters is

$$\beta = 1 + \frac{\nu - 3}{2(\eta - 1)}. \quad (4)$$

The nonlinear SDE (3) has the simplest form of the multiplicative noise term, $\sigma x^\eta dW$. Multiplicative equations with the drift coefficient proportional to the Stratonovich drift correction for transformation from the Stratonovich to the Itô stochastic equation [24] generate signals with the power-law distributions

[5]. Equation (3) is of such a type and has the stationary probability distribution of the power-law form, $P_0(x) \sim x^{-\nu}$.

Nonlinear SDE, corresponding to a particular case of (3) with $\eta = 0$, i.e., with linear noise and non-linear drift, was considered in Ref. [6]. It has been found that if the damping decrease with increasing of $|x|$, then the solution of such a nonlinear SDE has long correlation time.

Because of the divergence of the power-law distribution and the requirement of the stationarity of the process, the SDE (3) should be analyzed together with the appropriate restrictions of the diffusion in some finite interval. The simplest choice is the reflective boundary conditions at $x = x_{\min}$ and $x = x_{\max}$. However, other forms of restrictions are possible. For example, the exponential restriction of diffusion can be obtained by introducing the additional terms in (3),

$$dx = \sigma^2 \left(\eta - \frac{\nu}{2} + \frac{m}{2} \left(\frac{x_{\min}}{x} \right)^m - \frac{m}{2} \left(\frac{x}{x_{\max}} \right)^m \right) x^{2\eta-1} dt + \sigma x^\eta dW. \quad (5)$$

Here m is a parameter. Modified SDE

$$dx = \sigma^2(\eta - \nu/2)(x + x_0)^{2\eta-1} dt + \sigma(x + x_0)^\eta dW \quad (6)$$

with the reflective boundary condition at $x = 0$ was considered in [5]. The associated Fokker-Planck equation gives distribution density $P_0(x) \sim \exp_{1+1/\nu}(-\nu x/x_0)$, where $\exp_q(\cdot)$ is the q -exponential function defined as

$$\exp_q(x) \equiv (1 + (1 - q)x)^{\frac{1}{1-q}}. \quad (7)$$

The probability distributions containing q -exponential are important in the framework of the nonextensive statistical mechanics [25]–[28].

Stochastic differential equation, analyzed in [29]–[31]

$$dx = \sigma^2(\eta - \nu/2)(x^2 + x_0^2)^{\eta-1} x dt + \sigma(x^2 + x_0^2)^{\eta/2} dW, \quad (8)$$

in contrast to all other equations allows negative values of x . The associated Fokker-Planck equation gives the steady-state distribution density $P_0(x) \sim \exp_{1+2/\nu}(-\nu x^2/2x_0^2)$. The addition of parameter x_0 in (6), (8) restricts the divergence of the power-law distribution of x at $x \rightarrow 0$. For small $|x| \ll x_0$ equations (6), (8) represent the linear additive stochastic processes generating the Brownian motion with the steady drift, while for $x \gg x_0$ they reduce to the multiplicative equation (3).

A. Special cases

For some choices of parameters, SDE (3) or its variant (5) takes the form of the well-known SDEs considered in econophysics and finance.

In case when the exponent of multiplicative noise $\eta = 0$ and $\sigma = 1$, (3) takes the form of the SDE for the *Bessel process* [32],

$$dx = \frac{\delta - 1}{2} \frac{1}{x} dt + dW, \quad (9)$$

of dimension $\delta = 1 - \nu$, while $\eta = 1/2$, $\sigma = 2$ corresponds to the *squared Bessel process* [32],

$$dx = \delta dt + 2\sqrt{x} dW, \quad (10)$$

of dimension $\delta = 2(1 - \nu)$.

SDE with exponential restriction (5) for $\eta = 1/2$, $x_{\min} = 0$ and $m = 1$ gives the *Cox-Ingersoll-Ross* (CIR) process [32],

$$dx = k(\theta - x)dt + \sigma\sqrt{x}dW, \quad (11)$$

where $k = \sigma^2/2x_{\max}$, $\theta = x_{\max}(1 - \nu)$.

When $\nu = 2\eta$, $x_{\max} = \infty$ and $m = 2\eta - 2$ then (5) takes the form of the *Constant Elasticity of Variance* (CEV) process [32],

$$dx = \mu x dt + \sigma x^\eta dW, \quad (12)$$

where $\mu = \sigma^2(\eta - 1)x_{\min}^{2(\eta-1)}$.

III. CONNECTION OF $f^{-\beta}$ BEHAVIOR WITH EIGENVALUES OF THE FOKKER-PLANCK EQUATION

According to Wiener-Khintchine relations, the power spectral density is

$$S(f) = 2 \int_{-\infty}^{\infty} C(t) e^{i\omega t} dt = 4 \int_0^{\infty} C(t) \cos(\omega t) dt, \quad (13)$$

where $\omega = 2\pi f$. $C(t)$ is the autocorrelation function. For the stationary process the autocorrelation function can be expressed as an average over the realizations of the stochastic process, i.e., $C(t) = \langle x(t)x(t+t) \rangle$. This average can be written as

$$C(t) = \int dx \int dx' x x' P_0(x) P_x(x', t|x, 0), \quad (14)$$

where $P_0(x)$ is the steady-state probability distribution function and $P_x(x', t|x, 0)$ is the transition probability (the conditional probability that at time t the signal has value x' with the condition that at time $t = 0$ the signal has the value x). The transition probability can be obtained from the solution of the associated Fokker-Planck equation with the initial condition $P_x(x', 0|x, 0) = \delta(x' - x)$.

The solutions of the Fokker-Planck equation of the form $P(x, t) = P_\lambda(x) e^{-\lambda t}$ determine the eigenvalues λ and corresponding eigenfunctions $P_\lambda(x)$. It should be noted that the restriction of diffusion of the variable x by x_{\min} and x_{\max} ensures that the eigenvalue spectrum is discrete. Expansion of the transition probability density in a series of the eigenfunctions has the form [33]

$$P_x(x', t|x, 0) = \sum_{\lambda} P_\lambda(x') e^{\Phi(x')} P_\lambda(x) e^{-\lambda t}, \quad (15)$$

where $\Phi(x)$ is the potential, associated with the Fokker-Planck equation, $\Phi(x) = -\ln P_0(x)$. Substituting (15) into (14) we obtain the autocorrelation function

$$C(t) = \sum_{\lambda} e^{-\lambda t} X_{\lambda}^2. \quad (16)$$

Here

$$X_{\lambda} = \int_{x_{\min}}^{x_{\max}} x P_{\lambda}(x) dx \quad (17)$$

is the first moment of the stochastic variable x evaluated with the λ -th eigenfunction $P_{\lambda}(x)$. Such an expression for

the autocorrelation function has been obtained in Ref. [34]. From (13) and (16) we obtain the power spectral density

$$S(f) = 4 \sum_{\lambda} \frac{\lambda}{\lambda^2 + \omega^2} X_{\lambda}^2. \quad (18)$$

This expression for the power spectral density resembles the models of $1/f$ noise using the sum of the Lorentzian spectra [13], [35]–[40]. Here we see that the Lorentzians can arise from the single nonlinear SDE. Replacing in (18) the summation by the integration we obtain the power spectral density as

$$S(f) \approx 4 \int \frac{\lambda}{\lambda^2 + \omega^2} X_{\lambda}^2 D(\lambda) d\lambda. \quad (19)$$

Here $D(\lambda)$ is the density of eigenvalues. Equation, similar to (19) has been obtained in Ref. [41] by considering a relaxing linear system driven by the white noise. Similar equation has been obtained also in Ref. [42] where the reversible Markov chains on finite state spaces were considered. In both Ref. [41] and Ref. [42], the power spectral density is expressed as a sum or an integral over the eigenvalues of a matrix describing transitions in the system.

In Ref. [43] it has been shown that the largest contribution to the sum in (18) makes the terms corresponding to the eigenvalues λ obeying the condition $\lambda_{\min} \ll \lambda \ll \lambda_{\max}$, where

$$\lambda_{\min} = \sigma^2 x_{\min}^{2(\eta-1)}, \quad \lambda_{\max} = \sigma^2 x_{\max}^{2(\eta-1)}, \quad \eta > 1, \quad (20)$$

$$\lambda_{\min} = \sigma^2 x_{\max}^{2(\eta-1)}, \quad \lambda_{\max} = \sigma^2 x_{\min}^{2(\eta-1)}, \quad \eta < 1. \quad (21)$$

One of the reasons for the appearance of the $1/f^{\beta}$ spectrum is the scaling property of the SDE (3): changing the stochastic variable from x to $x' = ax$ changes the time-scale of the equation to $t' = a^{2(1-\eta)}t$, leaving the form of the equation invariant. From this property it follows that it is possible to eliminate λ in the eigenvalue equation by changing the variable from x to $z = \lambda^{1/2(1-\eta)}x$. The dependence of the eigenfunction on the eigenvalue λ then enters only via the boundary conditions. Such scaling properties were used in [43] estimating the expression $X_{\lambda}^2 D(\lambda)$ being proportional to $\lambda^{-\beta}$, with β given by (4). Thus the power spectral density of SDE (3) can be approximated as [43]

$$S(f) \sim \int_{\lambda_{\min}}^{\lambda_{\max}} \frac{1}{\lambda^{\beta-1}} \frac{1}{\lambda^2 + \omega^2} d\lambda. \quad (22)$$

When $\lambda_{\min} \ll \omega \ll \lambda_{\max}$, the leading term in the expansion of the approximate expression (22) for the power spectral density in the power series of ω is

$$S(f) \sim \omega^{-\beta}, \quad \beta < 2. \quad (23)$$

The second term in the expansion is proportional to ω^{-2} . In the case of $\beta < 2$, the term with $\omega^{-\beta}$ becomes larger than the term with ω^{-2} when $\lambda_{\min} \ll \omega$.

Therefore, we obtain $1/f^{\beta}$ spectrum in the frequency interval $\sigma^2 x_{\min}^{2(\eta-1)} \ll \omega \ll \sigma^2 x_{\max}^{2(\eta-1)}$ if $\eta > 1$ and the frequency interval $\sigma^2 x_{\max}^{-2(1-\eta)} \ll \omega \ll \sigma^2 x_{\min}^{-2(1-\eta)}$ if $\eta < 1$.

Equation (19) shows that the shape of the power spectrum depends on the behavior of the eigenfunctions and eigenvalues in terms of function $X_{\lambda}^2 D(\lambda)$. The SDE (3) considered in this article gives the density of eigenvalues $D(\lambda)$ proportional to $1/\sqrt{\lambda}$. One obtains $1/f^{\beta}$ behavior of the power spectrum when the function $X_{\lambda}^2 D(\lambda)$ is proportional to $\lambda^{-\beta}$ for a wide range of eigenvalues λ , as is the case for SDE (3). Similar condition has been obtained in Ref. [42].

IV. ANALYTICALLY SOLVABLE CASE

As an example we present the analytically solvable model, i.e., the particular CEV process. The choice of parameters $\eta = \frac{3}{2}$, $\nu = 3$ and $x_{\max} = \infty$ in (5) leads to the equation

$$dx = \mu x dt + \sigma x^{\frac{3}{2}} dW. \quad (24)$$

Here $\mu = \sigma^2 x_{\min}/2$. The analytical expression for the transition probability $P_x(x', t|x, 0)$ of the CEV process (24) is [32]

$$\begin{aligned} P_x(x', t|x, 0) &= \frac{x_{\min}}{(1 - e^{-\mu t})} \sqrt{\frac{x}{x'}} \\ &\times \exp\left(\frac{1}{2}\mu t - \frac{x_{\min}}{(1 - e^{-\mu t})} \left(\frac{1}{x'} + \frac{1}{x} e^{-\mu t}\right)\right) \\ &\times I_1\left(\frac{x_{\min}}{\sinh(\frac{1}{2}\mu t)} \frac{1}{\sqrt{xx'}}\right). \end{aligned} \quad (25)$$

Here I_n is the modified Bessel function with index n . The steady-state probability distribution has the form

$$P_0(x) = x_{\min}^2 x^{-3} \exp(-x_{\min}/x). \quad (26)$$

The average of the signal is $\langle x \rangle = \int_0^{\infty} x P_0(x) dx = x_{\min}$. From (14) using (25) we obtain the autocorrelation function

$$C(t) = -x_{\min}^2 e^{\mu t} \ln(1 - e^{-\mu t}). \quad (27)$$

When $\mu t \ll 1$ we get

$$C(t) \approx -x_{\min}^2 \ln(\mu t). \quad (28)$$

Similar expansions have been obtained for the autocorrelation function in the case of $1/f$ spectrum [5], [43]. From the Wiener-Khinchine relation (13), using (27) for the autocorrelation function, we get the following expression for the power spectral density

$$S(f) = 2x_{\min}^2 \left[\frac{-\gamma - \psi\left(-\frac{i\omega}{\mu}\right)}{\mu + i\omega} + \frac{-\gamma - \psi\left(\frac{i\omega}{\mu}\right)}{\mu - i\omega} \right], \quad (29)$$

where $\gamma \approx 0.577216$ is the Euler's constant and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function. When $\omega \gg \mu$ then the power spectral density is

$$S(f) \approx x_{\min}^2 / f. \quad (30)$$

We compare the analytical expressions for the steady-state probability distribution (26) and power spectral density (29) with those obtained from the numerical solution of (24). For the numerical solution we use the Euler-Marujama approximation, transforming differential equations to the difference

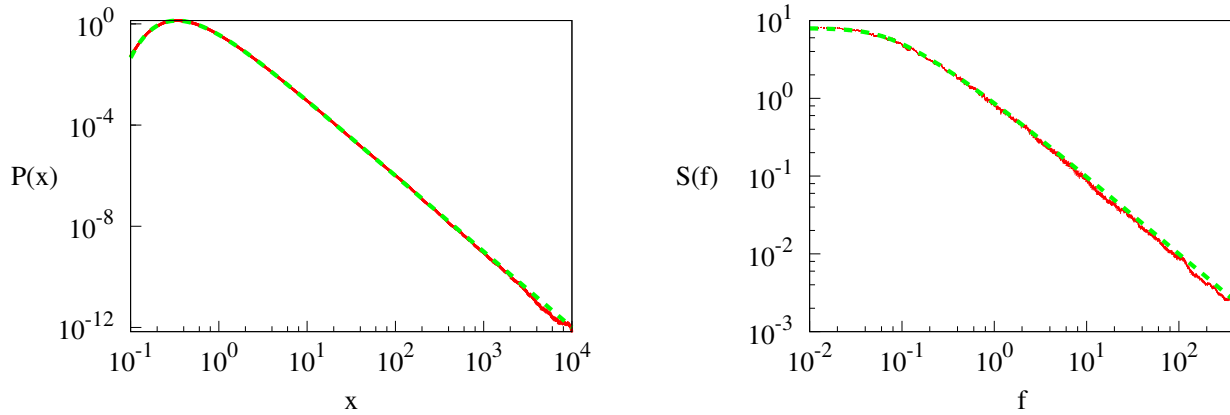


Fig. 1. Probability distribution function $P(x)$ (left) and power spectral density $S(f)$ (right) for the stochastic process defined by the SDE (24). Dashed green lines are analytical expressions (26) and (29) for the steady-state distribution function $P_0(x)$ and power spectral density $S(f)$, respectively. Parameters used are $\sigma = 1$ and $x_{\min} = 1$.

equations. The equation was solved using a variable step of integration, $\Delta t_k = \kappa^2/x_k$, with $\kappa \ll 1$ being a small parameter. The comparison of the analytical expressions with the numerical calculations is presented in Fig. 1. We see an excellent agreement between the analytical and numerical results.

Note, that in Ref. [44] asymptotic expansions of option prices for another CEV model, using the perturbation theory for the partial differential equations, are presented.

V. CONCLUSIONS

We considered a class of nonlinear SDEs, giving the power-law behavior of the power spectral density in any desirably wide range of frequency. The equations, as special cases, contain the well-known SDEs in economics and finance. The analysis reveals that the power spectrum may be represented as a sum of the Lorentzian spectra and provides further insights into the origin of $1/f$ noise.

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