Modeling the inverse cubic distributions by nonlinear stochastic differential equations

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***Abstract*—One of stylized facts emerging from statistical analysis of financial markets is the inverse cubic law for the cumulative distribution of a number of events of trades and of the logarithmic price change. A simple model, based on the point process model of $1/f$ noise, generating the long-range processes with the inverse cubic cumulative distribution is proposed and analyzed. Main assumptions of the model are proportional to the process intensity, $1/\tau(t)$, stochasticity of large interevent time $\tau(t)$ and the Brownian motion of small interevent time.**

I. Introduction

One of the principal statistical features characterizing the activity in financial markets is the distribution of fluctuations of market indicators such as the indexes. Frequently heavy-tailed long-range distributions with characteristic power-law exponents are observable. Power laws appear for relevant financial fluctuations, such as fluctuations of number of trades, trading volume and price. The well-identified stylized fact is the so-called inverse cubic power-law of the cumulative distributions, which is relevant to the developed stock markets, to the commodity one, as well as to the most traded currency exchange rates. The exponents that characterize these power laws are similar for different types and sizes of markets, for different market trends and even for different countries – suggesting that a generic theoretical basis may inspire these phenomena [1]–[12].

Here we propose a simple model, based on the point process model of $1/f$ noise [13]–[17] and the nonlinear stochastic differential equations [18]–[20] generating signals with $1/f^{\beta}$ ($0 \leq \beta \leq 2$) noise. The nonlinear stochastic differential equation for the interevent time $\tau(t)$ consists of a superposition of two processes: (i) the restricted additive Brownian motion in time of the interevent interval $\tau(t)$ for the frequent events (i.e., for small interevent time $\tau(t)$) and (ii) the multiplicative motion of the interevent time with the multiplicative noise proportional to the intensity, $1/\tau(t)$, of the process for the large interevent times $\tau(t)$. The proposed model generates long-range processes with two slopes of the power-law distribution, including the inverse cubic distribution, and the power-law distributions of power spectrum with two exponents. Analytical and numerical analysis of the proposed model is presented.

II. The model

Trades in the financial markets occur at the discrete times t_1, t_2, ..., t_k, ... and can be considered as a process of events. Such process is stochastic and defined by the stochastic interevent times $\tau_k \equiv t_{k+1} - t_k$. The generic multiplicative process for the interevent time [15], [16] and for the stochastic rate $x(t) = a/\tau(t)$ of events flow [21]–[23] has been introduced and analyzed.

The simplest stochastic model of the process of events is the point process [24]

$$x(t) = a \sum_k \delta(t - t_k) \tag{1}$$

representing the fluctuating variable $x(t)$ as consisting of a sequence of events at the discrete times $\{t_k\}$, Here $\delta(t)$ is the Dirac δ-function and a is an average contribution to the variable $x(t)$ of one event in the region of its occurrence time t_k. The low-frequency fluctuations of the long-range process are defined by the fluctuations and statistical properties of the time difference $t_{k+q} - t_k$ at large q, determined by the slow dynamics of the average interevent time $\tau_k(q) = (t_{k+q} - t_k)/q$ between the k-th and $(k+q)$-th events [16] .

Quite generally, the dependence of the average interevent time τ_k on the occurrence number k may be described by the general Langevin equation with the drift coefficient $d(\tau_k)$ and a multiplicative noise $b(\tau_k)\xi(k)$,

$$\frac{d\tau_k}{dt} = d(\tau_k) + b(\tau_k)\xi(k) . \tag{2}$$

Here $\xi(k)$ is the Gaussian white noise, $\left\langle \xi(k)\xi(k') \right\rangle = \delta(k - k')$, where the brackets $\langle ... \rangle$ indicate the average. Transition from the occurrence number k to the actual time t in (2) according to the relation $dt = \tau_k dk$ yields the Itô stochastic differential equation in the actual time t,

$$d\tau = \frac{d(\tau)}{\tau}dt + \frac{b(\tau)}{\sqrt{\tau}}dW , \tag{3}$$

where W is a standard Wiener process.

The generic multiplicative process generating the power-law distributed, $P_k(\tau_k) \sim \tau_k^{\alpha}$, sequence of the interevent time τ_k is [15]–[17]

$$\tau_{k+1} = \tau_k + \sigma^2 \left(\frac{\alpha}{2} + \mu\right) \tau_k^{2\mu - 1} + \sigma \tau_k^{\mu} \varepsilon_k . \tag{4}$$

In this model the (average) interevent time τ_k fluctuates due to the random perturbations by a sequence of uncorrelated normally distributed random variables $\{\varepsilon_k\}$ with the zero expectation and unit variance, μ is the index of the nonlinearity, and σ is standard deviation of the white noise. Some motivations for the model (1)-(4) were given in papers [13]–[17], [20]–[23], [25]–[27].

The power spectral density of the process (1) generated according to iterative equation (4) is $1/f^{\beta}$ in any desirable wide range of frequency f [15], [16], i.e.,

$$S(f) \sim \frac{1}{f^{\beta}}, \qquad \beta = 1 + \frac{\alpha}{(3-2\mu)}. \tag{5}$$

The distribution density of the signal (1) intensity is the power-law, as well [16], [20],

$$P(x) \sim x^{-\lambda}, \qquad \lambda = 3 + \alpha. \tag{6}$$

Therefore, for the pure $1/f$ noise, generated by the simplest iterative equations, e.g.,

$$\tau_{k+1} = \tau_k + \sigma\varepsilon_k, \tag{7}$$

with the appropriate boundary conditions, restricting the diffusion of τ_k in the finite interval [18] corresponds $P(x) \sim x^{-3}$ or *the inverse squared*, $P_>(x) \sim x^{-2}$, cumulative distribution.

The simplest equations generating *the inverse cubic law* of the cumulative distribution, $P_>(x) \sim x^{-3}$, are

$$\tau_{k+1} = \tau_k + \sigma\tau_k^{-1/2}\varepsilon_k, \tag{8}$$

$$d\tau(t) = \frac{\sigma}{\tau(t)}dW, \tag{9}$$

and

$$d\tau(t) = \sigma_x x(t)\, dW, \tag{10}$$

where $x(t) = a/\tau(t)$ and $\sigma_x = \sigma/a$.

Equation (10) reveals the particularly obvious meaning, i.e., *the intensity of fluctuations of the interevent time $\tau(t)$ is proportional to the intensity of the process $x(t) \propto 1/\tau(t)$.*

The condition of the stationarity of the process requires the appropriate restrictions of the movement of τ in some interval $[\tau_{\min}, \tau_{\max}]$. We will use the reflective boundary condition at small $\tau_k = \tau_{\min}$ and the exponential restriction of diffusion at large $\tau_k = \tau_{\max}$ by introducing the additional term in (8),

$$\tau_{k+1} = \tau_k - \sigma^2\frac{1}{\tau_{\max}} + \sigma\frac{1}{\sqrt{\tau_k}}\varepsilon_k. \tag{11}$$

The steady-state distribution density $P_k(\tau_k)$ in k-space of interevent time τ_k, given from the associated Fokker-Planck equation of process (11) is [16], [28], [29]

$$P_k(\tau_k) \simeq \frac{2\tau_k}{\tau_{\max}^2}\exp\left(-\frac{\tau_k^2}{\tau_{\max}^2}\right). \tag{12}$$

Consequently, the steady-state distribution of the intensity of the process $x(t) = 1/\tau(t)$, exponentially restricted at small $x = x_{\min} = 1/\tau_{\max}$ is (see [16], [19] for analogy)

$$P(x) \simeq \frac{4x_{\min}^3}{\sqrt{\pi}x^4}\exp\left(-\frac{x_{\min}^2}{x^2}\right). \tag{13}$$

Here and further for the brevity we continue the analysis for $a = 1$. The generalization for $a \neq 1$ is straightforward.

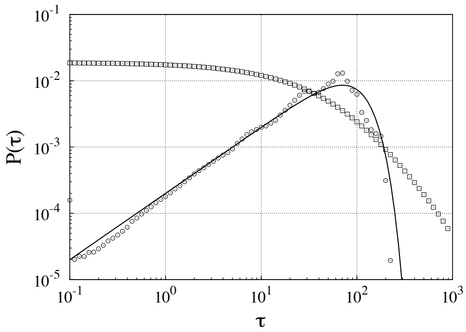

Fig. 1. The steady-state distribution density $P_k(\tau_k)$ of interevent time τ_k calculated according to (11), open circles, and the Poissonian-like distribution density $P_j(\tau_j)$ converted from τ_k by (16), open squares. Used parameters are $\tau_{\min} = 0.1$, $\tau_{\max} = 100$, and $\sigma = 0.1$. Solid line represents the analytical result (12).

The cumulative distribution $P_>(x)$ of x is

$$\begin{aligned} P_>(x) &= \int_x^{\infty} P(x)\, dx \\ &\simeq \operatorname{erf}\left(\frac{x_{\min}}{x}\right) - \frac{2x_{\min}}{\sqrt{\pi}x}\exp\left(-\frac{x_{\min}^2}{x^2}\right) \\ &= \frac{x_{\min}^3}{x^3}\gamma^*\left(\frac{3}{2}, \frac{x_{\min}^2}{x^2}\right). \end{aligned} \tag{14}$$

Here $\gamma^*(a, z)$ is the regularized lower incomplete gamma function. Consequently

$$P_>(x) \simeq \frac{4x_{\min}^3}{3\sqrt{\pi}x^3}, \qquad x \gg x_{\min}, \tag{15}$$

and we find out *the inverse cubic law*.

A. Poissonian-like process

Further we can consider a more realistic model assuming that τ_k is a time-dependent average interevent time of the Poissonian-like process with the time-dependent rate. Within this assumption the actual interevent time τ_j is given by the conditional probability [17], [22]

$$\varphi(\tau_j|\tau_k) = \frac{1}{\tau_k}e^{-\tau_j/\tau_k}, \tag{16}$$

similar to the non-homogeneous Poisson process. In such a case, the distribution of the actual interevent time τ_j is expressed analogically to the superstatistical schemes [30],

$$P_j(\tau_j) = \int \varphi(\tau_j|\tau_k)P_k(\tau_k)d\tau_k. \tag{17}$$

The generalized model (16) and (17) represents a more realistic situation, because the concrete event occurs at random time (like in the Poisson case), however, the average interevent time is slowly (Brownian-like) modulated.

This additional stochasticity of the actual interevent time τ_j by randomization (16) of the concrete occurrence times does not influence on the low frequency power spectra of the signal.

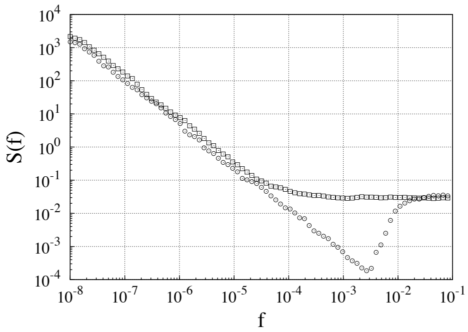

Fig. 2. Power spectral density $S(f)$ of the signal $x(t)$ (1) calculated according to Eq. (11), open circles, and that of the Poissonian-like distributed (16) interevent time τ_j, open squares. Used parameters are as in Fig. 1.

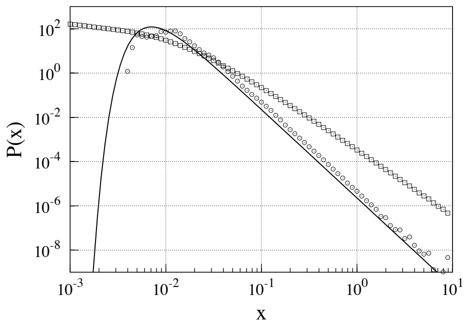

Fig. 3. Probability distribution density $P(x)$ of the signal (1). Used parameters and notations are as in Fig. 1. Solid line represents the analytical result (13).

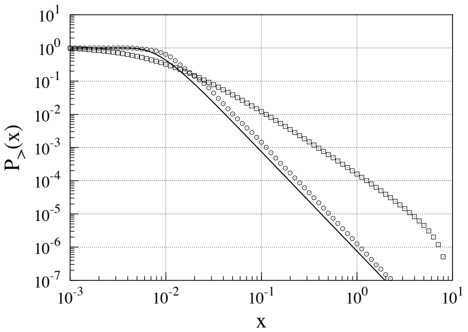

Fig. 4. Cumulative distribution $P_>(x)$ of the signal (1). Used parameters and notations are as in Fig. 1. Solid line represents the analytical result (14).

B. Numerical analysis

The numerical calculations of the distribution of the interevent time, $P_k(\tau_k)$, power spectral density, $S(f)$, distribution density of the signal, $P(x)$, and the cumulative distribution, $P_>(x)$, are presented in Fig. 1, Fig. 2, Fig. 3, and Fig. 4, respectively. We observe in Fig. 1 the analytically predicted (12) distribution of τ_k and the Poissonian-like distribution density $P_j(\tau_j) \simeq cont$ for $\tau_j \ll \tau_{\max}$, as given in Ref. [17]. As far as in this case $\mu = -1/2$ and $\alpha = 1$, according to (5), $\beta = 5/4$, i.e., at low frequency in agreement with Fig. 2, both without and with the additional stochasticity (16).

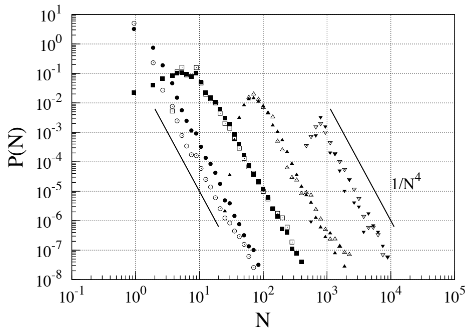

Fig. 5. Probability distribution function $P(N)$ of the counting of events (18) for the process (11) without, open symbols, and with, full symbols, the additional Poissonian stochasticity (16). Used parameters are as in Fig. 1. Results are presented for $\langle N \rangle = 1$, 10, 100, and 1000, the increasing probabilities, respectively.

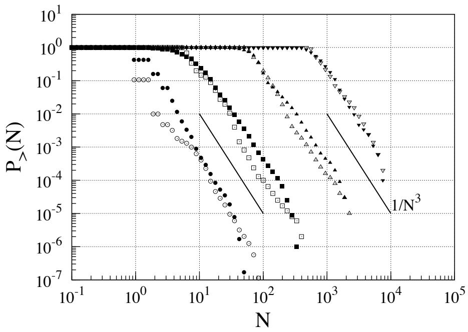

Fig. 6. Cumulative distribution $P_>(N)$ of the counting of events (18). Used parameters and notations are as in Fig. 5

Figs. 3 and 4 demonstrate the distribution density and *the inverse cubic* cumulative distribution of the signal (1) without the additional Poissonian stochasticity (16). This extra randomization (16) results in the flat distribution, $P_j(\tau_j) \simeq cont$, of the actual interevent time τ_j and, consequently, in the inverse squared cumulative distribution, as it follows from (6) for $\alpha = 0$.

Variable $x(t) = 1/\tau(t)$ represents the formal instantaneous process and does not contain any scale of time. Actually, one measures the number of events N in the definite time window τ_w, e.g., the trading activity, as a number of events in some time interval, or the return at time lag τ_w. These quantities are represented as the integral of the variable $x(t)$ in time interval τ_w or counting of events in the time lag τ_w [15], [21]–[23],

$$N(t) = \int_{t}^{t+\tau_w} x(t')\, dt'. \tag{18}$$

Figs. 5 and 6 demonstrate the distribution density of N and *the inverse cubic* cumulative distribution of N both without and with the additional Poissonian stochasticity (16).

III. Generalization of the model

For modeling the long-range processes with $\beta < 1$ and with the power-law correlation function [20]

$$C(t) \sim \frac{1}{t^{1-\beta}} \tag{19}$$

we should modify Eqs. (8)-(10) assuming the simple additive Brownian motion of small interevent time, keeping the same dependence for large $\tau(t)$. For this purpose, instead of (9) we propose equation

$$d\tau = \sigma \frac{1}{\tau_c + \tau} dW\,, \tag{20}$$

where τ_c is a crossover parameter, separating the two kinds of the stochastic motion: (i) the simple Brownian motion for $\tau \ll \tau_c$ and (ii) the model of Section II for $\tau \gg \tau_c$.

Eq. (20) with restrictions at $\tau = \tau_{\min}$ and at $\tau = \tau_{\max}$

$$d\tau = \sigma^2 \left(\frac{\tau_{\min}^2}{\tau^2} - \frac{\tau^2}{\tau_{\max}^2} \right) \frac{dt}{\tau\left(\tau_c + \tau\right)^2} + \sigma \frac{dW}{\tau_c + \tau}\,. \tag{21}$$

may be solved using a variable step of integration

$$\Delta t_i = \frac{\kappa^2}{\sigma^2} \left(\tau_c + \tau_i\right)^2 \tau_i^2\,,\; \kappa \ll 1, \tag{22}$$

$$\tau_{i+1} = \tau_i + \kappa^2 \left(\frac{\tau_{\min}^2}{\tau_i^2} - \frac{\tau_i^2}{\tau_{\max}^2} \right) \tau_i + \kappa \tau_i \varepsilon_i\,. \tag{23}$$

Note, that iterative equations in k-space like (4), (7), (8), and (11) correspond to the integration step $\Delta t_k = \tau_k$.

The steady-state distribution density $P_k(\tau_k)$ in k -space of interevent time τ_k, instead of (12), for $\tau_{\min} \ll \tau_c \ll \tau_{\max}$ is

$$P_k(\tau_k) \simeq \frac{2\left(\tau_c + \tau_k\right)^2}{\tau_{\max}^2 \tau_k} \exp\left(-\frac{\tau_{\min}^2}{\tau_k^2} - \frac{\tau_k^2}{\tau_{\max}^2} \right). \tag{24}$$

The steady-state distribution of the intensity of the process $x(t)$, exponentially restricted at small $x_{\min} = 1/\tau_{\max}$ and large $x_{\max} = 1/\tau_{\min}$, is

$$P(x) \simeq \frac{4x_{\min}^3 \left(x_c + x\right)^2}{\sqrt{\pi} x^4} \exp\left(-\frac{x_{\min}^2}{x^2} - \frac{x^2}{x_{\max}^2} \right)\,. \tag{25}$$

The cumulative distribution $P_>(x)$ of x for $x < x_c$ is given by the same Eq. (14). The average intensity of the process $\langle x \rangle = \langle \tau_k \rangle^{-1}$, where $\langle \tau_k \rangle \simeq \frac{\sqrt{\pi}}{2}\tau_{\max}$. The counting of events may be calculated according to the same Eq. (18).

The numerical calculations of the power spectral density $S(f)$ of the signal $x(t)$ (1) calculated according to Eqs. (21)–(23) are presented in Fig. 7. The cumulative distributions of this generalization are similar to those of Fig. 4 and Fig. 6.

More complex equations for modeling the financial systems have been introduced and analyzed in Refs. [21]–[23], [27].

IV. Conclusions

Simple stochastic nonlinear differential equations generating the long-range processes with the inverse cubic cumulative distributions are proposed.

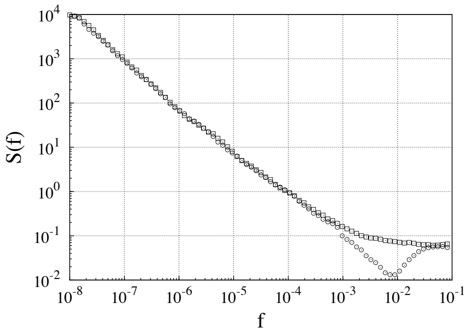

Fig. 7. Power spectral density $S(f)$ of the signal $x(t)$ (1) calculated according to Eqs. (21)–(23), open circles, and that of the Poissonian-like distributed (16) interevent time τ_j, open squares. Used parameters are $\tau_{\min} = 0.01$, $\tau_{\max} = 100$, $\tau_c = 1$, and $\sigma = \kappa = 0.1$.

Main assumptions of the model are: (i) the Brownian motion of small interevent time and (ii) the multiplicative stochasticity, proportional to the intensity of the process, of large interevent time.

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