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SYNCHRONIZATION OF CHAOTIC SYSTEMS DRIVEN BY IDENTICAL NOISE

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An analysis of transition from chaotic to nonchaotic behavior and synchronization in an ensemble of systems driven by identical random forces is presented. The synchronization phenomenon is investigated in the ensemble of particles moving with friction in the time-dependent potential and driven by the identical noise. The threshold values of the parameters for transition from chaotic to nonchaotic behavior are obtained and dependencies of the Lyapunov exponents and power spectral density of the current of the ensemble of particles on the nonlinearity of the systems and intensity of the driven force are analyzed.

1. Introduction

Often trajectories of the nonlinear dynamical systems are very sensitive to the initial conditions and unpredictable, i.e. the systems are chaotic. These systems exhibit an apparent random behavior. It might be expected that when turning on additional random forces make their behavior even "more chaotic". However, a transition from chaotic to nonchaotic behavior in an ensemble of particles with different initial conditions bounded in a fixed external potential and driven by an identical sequence of random forces was observed by Fahy and Hamann [1992] and recently analyzed theoretically and numerically by Kaulakys and Vektaris [1995a, 1995b] and Chen [1996]. It has been shown that the ensemble of trajectories in such a case may become identical at long times. The system becomes not chaotic: The trajectories are independent on the initial conditions. The similar effects have been observed in the different systems as well [Yu *et al.*, 1990; Maritan & Banavar, 1994a] and have resulted to same discussion concerning the origin and causality of such nonchaotic behavior [Pikovsky, 1994; Maritan & Banavar, 1994b; Gade & Basu, 1996].

Moreover, the observed effect resembles a phase transition but does not depend crucially on the dimension of the space in which the particles move. This phenomenon has some importance for Monte Carlo simulations and can influence on the clustering of particles process.

It should be noted that Maritan and Banavar [1994a] have analyzed the similar effect using the Langevin equation, however, in the limit in which the time separation between successive forces become small compared to any characteristic macroscopic time of the system and two logistic maps linked with a common noise term.

Here we analyze the similar phenomenon in the ensemble of particles moving with friction in the time-dependent potential and driven by the discrete identical noise. We define the threshold values of the parameters for transition from chaotic to nonchaotic behavior and investigate dependencies of the Lyapunov exponents and power spectral density on the nonlinearity of the systems and character of the driven force. Our theoretical analysis is based on the mapping form of equations of motion for the distance between the particles and the difference of the velocity of the particles while numerical calculations are performed according to the derived mapping equations as well as directly calculating the system's trajectories and the Lyapunov exponents. The mapping analysis results in the conclusions very close to those obtained from the direct simulations and numerical calculations

2. Models and Theory

Consider a system of particles of mass m moving with friction according to Newton's equations

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}\,, \quad \frac{d\mathbf{v}}{dt} = -\frac{1}{m}\frac{dV(\mathbf{r},\, t)}{d\mathbf{r}} - \gamma\mathbf{v} \tag{1}$$

in the time dependent potential $V(\mathbf{r},\, t)$, e.g. in the potential $V(x,\, t) = x^4 - x^2 - ax\sin\omega t$, and with the friction coefficient γ.

At time intervals τ the particles are partially stopped and their velocities are reset to the mixture of some part α of the old velocities with the random velocity $\mathbf{v}_k^{\mathrm{ran}}$: $\mathbf{v}^{\mathrm{new}} = \alpha\mathbf{v}^{\mathrm{old}} + \mathbf{v}_k^{\mathrm{ran}}$, where k is the stop number. Note that $\mathbf{v}_k^{\mathrm{ran}}$ depends on the stop number k but not on the particle. The simplest and most natural way is to choose the random values of velocity $\mathbf{v}_k^{\mathrm{ran}}$ from a Maxwell distribution with $k_BT = m = 1$, i.e. from the Gaussian distribution of variance $\sigma^2 = 1$. Figure 1 illustrates the difference of evolution of the ensemble of particles with randomly distributed (from the Gaussian distribution of variance $\sigma^2 = 1$) initial conditions and perturbed by the replacement $\mathbf{v}^{\mathrm{new}} = \alpha\mathbf{v}^{\mathrm{old}} + \mathbf{v}_k^{\mathrm{ran}}$ for different values of the time interval τ between such perturbations. We see the transition to one (common for all particles) trajectory for sufficiently small time interval τ.

Theoretically a transition from chaotic to nonchaotic behavior in such a system may be detected from analysis of the neighboring trajectories of two particles initially at points $\mathbf{r}_0$ and $\mathbf{r}_0'$ with starting velocities $\mathbf{v}_0$ and $\mathbf{v}_0'$. The convergence of the two trajectories to the single final trajectory depends on the evolution with a time of the small variances $\Delta\mathbf{r}_k = \mathbf{r}_k' - \mathbf{r}_k$ and $\Delta\mathbf{v}_k = \mathbf{v}_k' - \mathbf{v}_k$. From formal solutions $\mathbf{r} = \mathbf{r}(\mathbf{r}_k,\, \mathbf{v}_k,\, t)$ and $\mathbf{v} = \mathbf{v}(\mathbf{r}_k,\, \mathbf{v}_k,\, t)$ of the Newton's equations with initial conditions $\mathbf{r} = \mathbf{r}_k$ and $\mathbf{v} = \mathbf{v}_k$ at $t = 0$ it follows the mapping form of the equations of motion for $\Delta\mathbf{r}$ and $\Delta\mathbf{v}$ [Kaulakys & Vektaris, 1995a, 1995b]:

$$\begin{pmatrix} \Delta\mathbf{r}_{k+1} \\ \Delta\mathbf{v}_{k+1} \end{pmatrix} = \mathbf{T}(\alpha;\, \mathbf{r}_k,\, \mathbf{v}_k,\, \tau_k) \begin{pmatrix} \Delta\mathbf{r}_k \\ \Delta\mathbf{v}_k \end{pmatrix} \tag{2}$$

where the $\mathbf{T}$ matrix is of the form

$$\mathbf{T} = \begin{pmatrix} T_{\mathbf{rr}} & \alpha T_{\mathbf{rv}} \\ T_{\mathbf{vr}} & \alpha T_{\mathbf{vv}} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial\mathbf{r}}{\partial\mathbf{r}_k} & \alpha\dfrac{\partial\mathbf{r}}{\partial\mathbf{v}_k} \\[2ex] \dfrac{\partial\mathbf{v}}{\partial\mathbf{r}_k} & \alpha\dfrac{\partial\mathbf{v}}{\partial\mathbf{v}_k} \end{pmatrix} \tag{3}$$

and the time interval τ_k may depend on the step k.

For one-, two- and three-dimensional systems the dimension of the $\mathbf{T}$ matrix is 2, 4 and 6, respectively.

Matrix elements $T_{\mathbf{rr}}$ and $T_{\mathbf{rv}}$ satisfy the equation

$$\frac{d^2T_{\mathbf{r}}}{dt^2} = -\frac{1}{m}\left(T_{\mathbf{r}} \cdot \frac{d}{d\mathbf{r}}\right)\frac{dV(\mathbf{r},\, t)}{d\mathbf{r}}\bigg|_{\mathbf{r}=\mathbf{r}(\mathbf{r}_k,\, \mathbf{v}_k,\, t)} - \gamma\frac{dT_{\mathbf{r}}}{dt} \tag{4}$$

and initial conditions at $t = 0$

$$\begin{gathered} T_{\mathbf{rr}}(\mathbf{r}_k,\, \mathbf{v}_k,\, 0) = T_{\mathbf{vv}} = 1\,, \quad T_{\mathbf{rv}} = T_{\mathbf{vr}} = 0 \\ \dot{T}_{\mathbf{rr}}(\mathbf{r}_k,\, \mathbf{v}_k,\, 0) = \dot{T}_{\mathbf{vv}} = 0\,, \quad \dot{T}_{\mathbf{rv}} = 1\,, \\ \dot{T}_{\mathbf{vr}} = -\frac{1}{m}\frac{d^2V}{d\mathbf{r}^2}\bigg|_{x=x_k}, \end{gathered} \tag{5}$$

while $T_{\mathbf{vr}} = \dot{T}_{\mathbf{rr}}$ and $T_{\mathbf{vv}} = \dot{T}_{\mathbf{rv}}$. Here and further the points over the letters express the derivatives with respect to the time.

Further analysis is based on the general theory of dynamics of classical systems represented as maps: We can calculate the eigenvalues μ_k of the $\mathbf{T}$ matrix for each step and evaluate the averaged Lyapunov exponents or KS entropy of the system

$$\begin{aligned} \sigma_k &= \left\langle \frac{1}{\tau_k}\ln|\mu_k| \right\rangle \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\frac{1}{\tau_k}\ln|\mu_k(\mathbf{r}_k,\, \mathbf{v}_k,\, \tau_k)| \end{aligned} \tag{6}$$

A criterion for transition to chaotic behavior is

$$\sigma_{\max} = 0\,. \tag{7}$$

Comparisons of the threshold values τ_c for transition to chaos according to Eqs. (2)–(7) with those from the direct numerical simulations

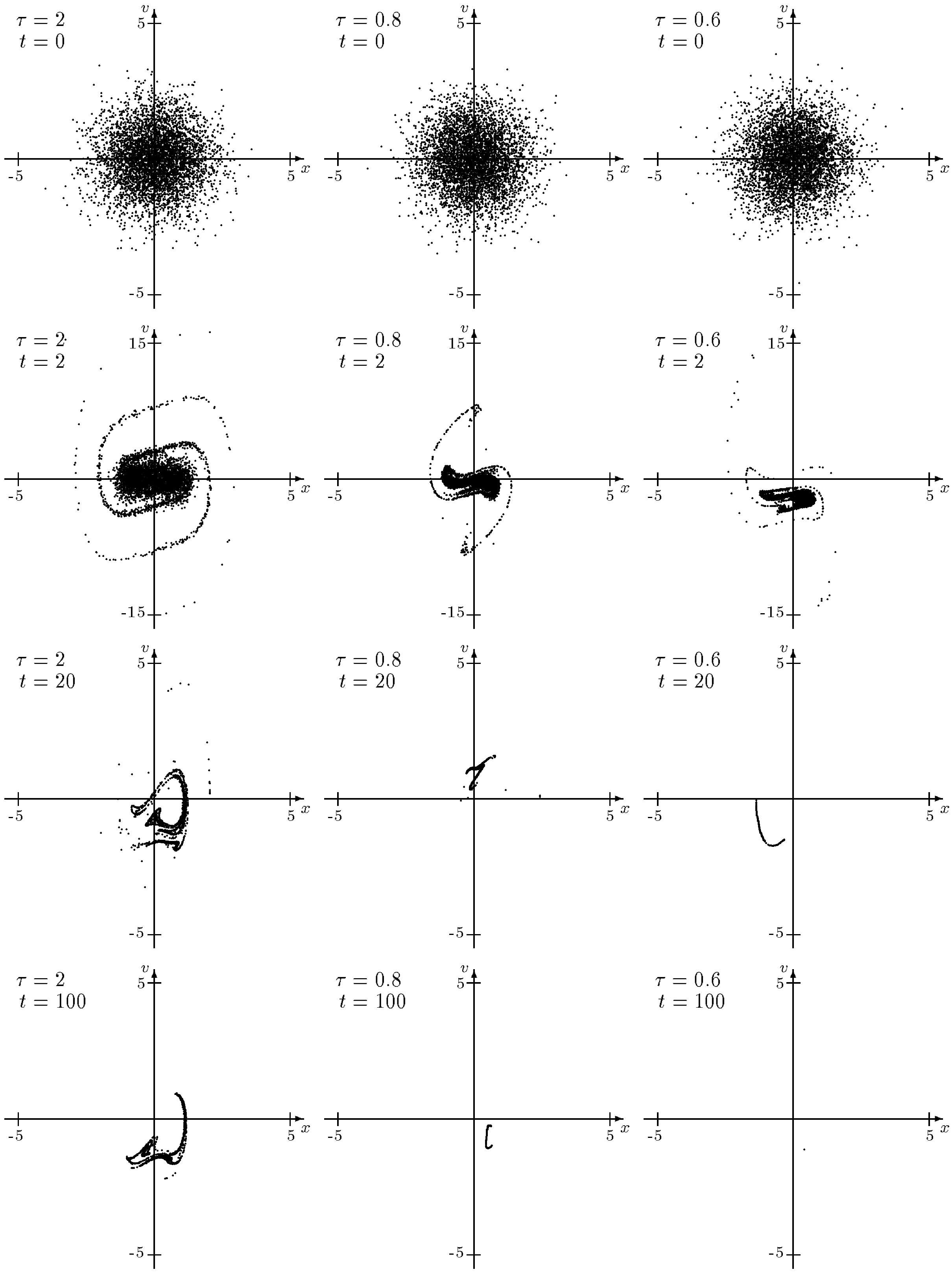

Fig. 1. Illustration of the difference of evolution in the $(x,\, v)$-space of the ensemble of particles in the autonomous Duffing potential $V(x) = x^4 - x^2$ with randomly distributed initial conditions and perturbed at time intervals τ by the replacement $v^{\text{new}}(k\tau) = \alpha v^{\text{old}}(k\tau) + v_k^{\text{ran}}$, $k = 1, 2, \ldots$ with $\alpha = 0.5$. For the relatively large $\tau = 2$ there is no transition to the common trajectory, for smaller $\tau = 0.8$ the clustering process of particles with different initial conditions is relatively slow while for sufficiently small $\tau = 0.6$ a collapse to the common trajectory at the time moment $t = 100$ is evident.

indicate to the fitness and usefulness of the method (2)–(7) for investigation of transition from chaotic to nonchaotic behavior in randomly driven ensemble of systems bounded in the fixed external potential without the friction [Kaulakys & Vektaris, 1995a, 1995b].

3. Results of Calculations

Here we calculate the Lyapunov exponents directly from the equations of motion and linearized equations for the variances of coordinate and velocity. Further we extend the same analysis for the systems with friction in the regularly time depending external field and perturbed by the identical for all particles random force.

Figures 2–5 represent an extensive analysis of the autonomous system based on the numerical solutions of the differential equations of motion. Figure 2 shows quite similar behavior of the Lyapunov exponent to that calculated from the mapping equations of motion (curve (a) of Fig. 2 in the paper [Kaulakys & Vektaris, 1995a]). Figures 3–5 represent dependences of the Lyapunov exponents on the different parameters of the model. Areas of the parameters for which the Lyapunov exponents are negative corresponds to the nonchaotic Brownian-type motion.

In general, motion in the nonautonomous Duffing potential with friction describe equations

$$\dot{v} = 2x - 4x^3 - \gamma v + a \sin \omega t\,, \quad \dot{x} = v\,. \tag{8}$$

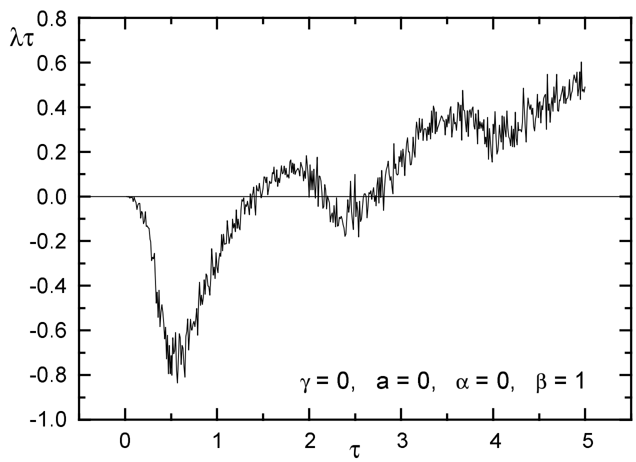

Fig. 2. Lyapunov exponent (multiplied by τ) from the direct calculations versus the time τ between the resets of the velocity $v^{\rm new}(k\tau) = \alpha v^{\rm old}(k\tau) + \beta v_k^{\rm ran}$, $k = 1, 2, \ldots$ for motion in the autonomous Duffing potential.

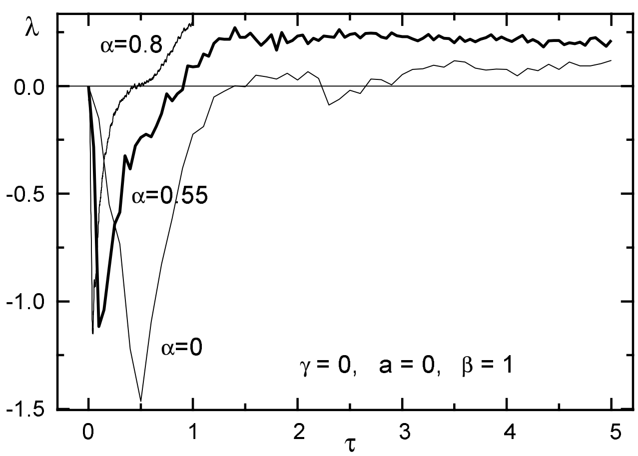

Fig. 3. Lyapunov exponent λ for motion in the autonomous Duffing potential versus the time interval τ between the resets of the velocity $v^{\rm new}(k\tau) = \alpha v^{\rm old}(k\tau) + \beta v_k^{\rm ran}$, $k = 1, 2, \ldots$ for different values of the parameter α.

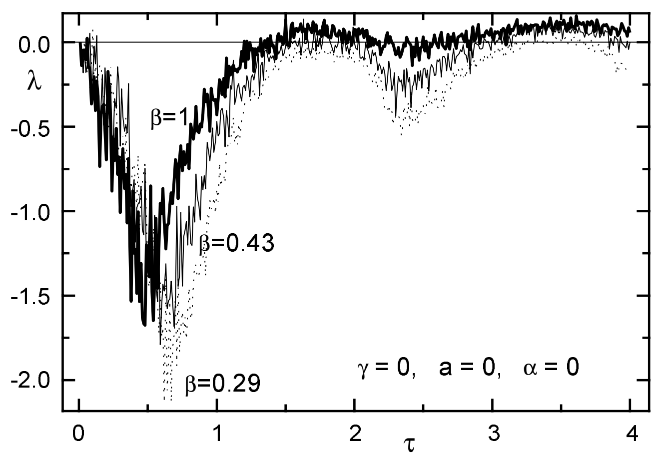

Fig. 4. As in Fig. 3 but for different values of β.

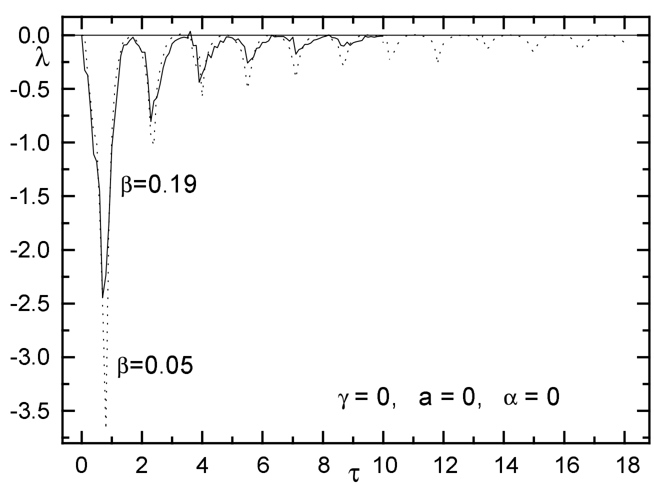

Fig. 5. As in Fig. 3 but for small values of β.

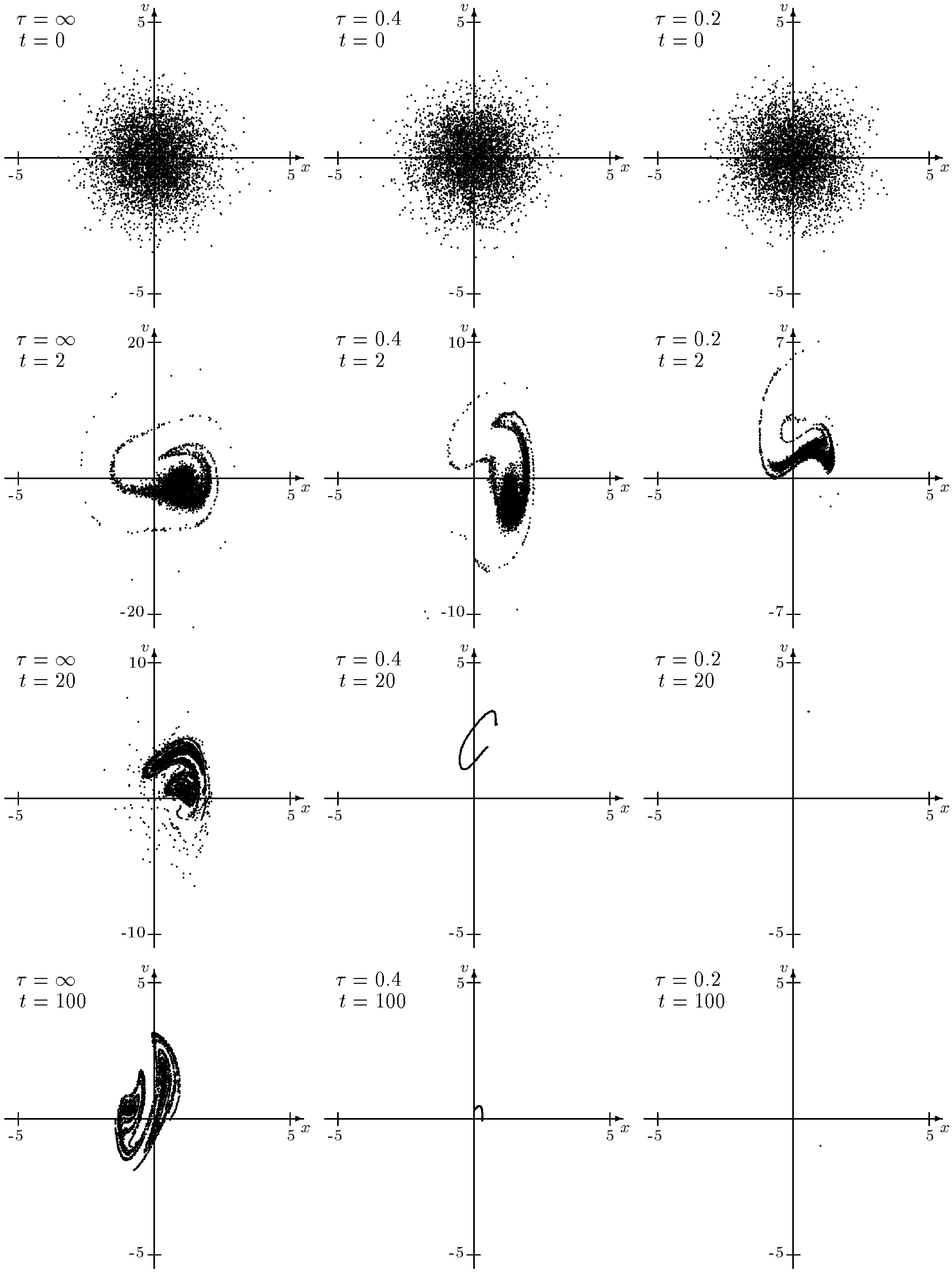

Fig. 6. As in the Fig. 1 but for motion according to Eq. (8) in the nonautonomous Duffing potential with $\gamma = 0.07$, $a = 5$, $\alpha = 0.8$ and $\beta = 1$. A transition from the actual chaotic (at $\tau = \infty$) to the nonchaotic dynamics with the decrease of the time interval τ between the resets of the velocity is observable.

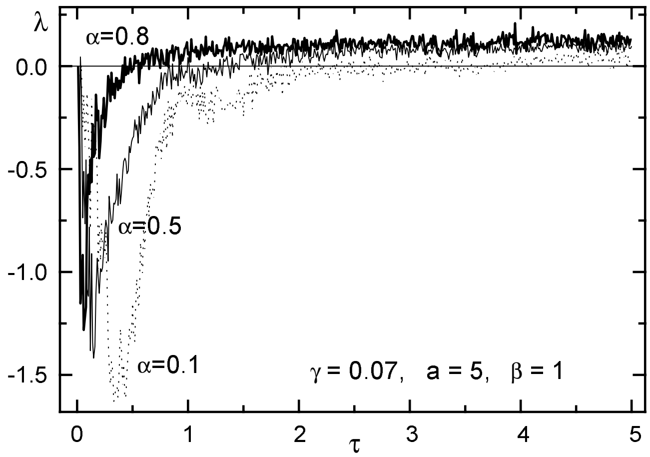

Fig. 7. Lyapunov exponent λ versus the time τ between the resets of the velocity $v^{\rm new}(k\tau) = \alpha v^{\rm old}(k\tau) + \beta v_k^{\rm ran}$, $k = 1, 2, \ldots$ for different values of the parameter α for motion in the driven Duffing potential with friction according to Eq. (8).

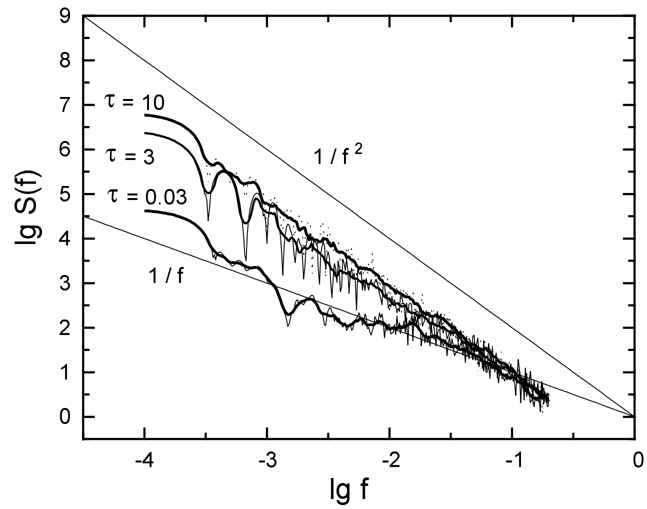

Fig. 8. The power spectral density of the current of the ensemble of particles moving according to Eq. (9) with $F = 1$, $\gamma = 0.1$ and perturbed by the common for all particles noise $v^{\rm new}(k\tau) = \alpha v^{\rm old}(k\tau) + v_k^{\rm ran}$, $k = 1, 2, \ldots$ with $\alpha = 1$ and different values of τ. The dense lines represent the averaged spectra.

For $a = 0$ and $\gamma = 0$ Eqs. (8) represent motion in the fixed external potential. As it was mentioned above, in the paper by Kaulakys and Vektaris [1995a] some theoretical and numerical analysis of this model was fulfilled on the bases of the mapping equations (2)–(7).

Figure 6 illustrates evolution of the ensemble of particles with randomly distributed initial conditions in the nonautonomous Duffing potential with friction and perturbed by the replacement $v^{\rm new}(k\tau) = \alpha v^{\rm old}(k\tau) + v_k^{\rm ran}$, $k = 1, 2, \ldots$. We observe a transition from the actual chaotic dynamics for large τ to the nonchaotic common for all particles trajectory with the decrease of τ. In Fig. 7 we show the dependence on τ of the Lyapunov exponents for the motion in the nonautonomous Duffing potential with friction. For the values of parameters corresponding to the positive Lyapunov exponents, i.e. without the random perturbation ($\tau \to \infty$), the system is chaotic. The negative Lyapunov exponents for small τ indicate to the nonchaotic Brownian-type motion.

4. Spectrum of the Current Noise

As it has already been observed in the paper by Kaulakys and Vektaris [1995b] such systems exhibit the intermittency route to chaos which provides sufficiently universal mechanism for $1/f$-type noise in the nonlinear systems. Here we analyze numerically the power spectral density of the current of the ensemble of particles moving in the closed contour and perturbed by the common for all particles noise. The simplest equations of motion for such model are of the form

$$\dot{v} = F - \gamma v\,, \quad \dot{x} = v \qquad (9)$$

with the perturbation given by the resets of velocity of all particles after every time interval τ according to the identical for all particles replacement $v^{\rm new}(k\tau) = \alpha v^{\rm old}(k\tau) + v_k^{\rm ran}$, $k = 1, 2, \ldots$. For sufficiently small τ we observe the current power spectral density $S(f)$ dependence on the frequency f close to the $1/f$-dependence (Fig. 8). It should be noted that such spectral density dependence is nonsensitive to some additional (nonlinear) terms in the equation for velocity. The essential condition for the $1/f$-type dependence of the current power spectral density is the random sufficiently strong perturbation of the particles' velocities (see also [Kaulakys & Meškauskas, 1997] for analysis of other systems and different perturbations).

5. Conclusions

From the fulfilled analysis we may conclude that, first, synchronization and transition from chaotic to nonchaotic behavior in ensembles of the identically perturbed by the random force nonlinear systems may be analyzed as from the mapping form of equations of motion for the distance between the particles and the difference of the velocity as well as from the direct calculations of the Lyapunov exponents and, second, the model of Fahy and Hamann [1992] may be generalized for the ensemble of particles moving with friction in the time-dependent potential.

The transition from chaotic to nonchaotic behavior in the ensemble of particles moving with friction may also be observed in the more realistic case of motion without the momently stops of the particles. On the other hand, the motion in the time-dependent potential without the random periodic perturbations even in the one-dimensional case may be chaotic or nonchaotic, depending on the system's parameters. Therefore, such generalization of the model allows to investigate the synchronization phenomenon and transition from chaotic to nonchaotic behavior effect in an ensemble of systems driven by identical random forces on the base of relatively simple (one-dimensional) models for larger variety of the system's dynamics.

Moreover, an ensemble of systems linked with a common external noise may exhibit the $1/f$-type fluctuations. Our model may easily be generalized for systems driven by any random forces or fluctuations. On the other hand, the phenomenon when an ensemble of systems is linked with a common external noise or fluctuating external fields is quite usual. Thus, an ensemble of systems in the external random field may provide a sufficiently universal mechanism of $1/f$-noise.

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