Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Modeling scaled processes and $1/f^{\beta}$ noise using nonlinear stochastic differential equations

B Kaulakys and M Alaburda

Institute of Theoretical Physics and Astronomy of Vilnius University, Goštauto 12, LT-01108 Vilnius, Lithuania
E-mail: kaulakys@itpa.lt and miglius@itpa.lt

Received 20 November 2008
Accepted 20 January 2009
Published 20 February 2009

Online at stacks.iop.org/JSTAT/2009/P02051
doi:10.1088/1742-5468/2009/02/P02051

Abstract. We present and analyze stochastic nonlinear differential equations generating signals with the power-law distributions of the signal intensity, $1/f^{\beta}$ noise, power-law autocorrelations and second-order structural (height–height correlation) functions. Analytical expressions for such characteristics are derived and a comparison with numerical calculations is presented. The numerical calculations reveal links between the proposed model and models where signals consist of bursts characterized by power-law distributions of burst size, burst duration and interburst time, as in the case of avalanches in self-organized critical models and the extreme event return times in long-term memory processes. The approach presented may be useful for modeling long-range scaled processes exhibiting $1/f$ noise and power-law distributions.

Keywords: stochastic processes (theory), stochastic processes, current fluctuations

©2009 IOP Publishing Ltd and SISSA
1742-5468/09/P02051+16$30.00

Contents

J. Stat. Mech. (2009) P02051

1. Introduction

Inverse power-law distributions, autocorrelations and spectra of signals, including $1/f$ noise (also known as $1/f$ fluctuations, flicker noise and pink noise), as well as scaling behavior in general, are ubiquitous in physics and in many other fields, including natural phenomena, spatial repartition of faults in geology, human activities such as the traffic in computer networks and financial markets. This subject has been a hot research topic for many decades (see, e.g., [1]–[10] and references herein). An up-to-date bibliographic list on $1/f$ noise of more than 1300 papers was composed by Li [11].

The widespread occurrence of signals exhibiting such a behavior suggests that a generic, at least mathematical explanation of the power-law distributions might exist. Note that the origins of two popular noises, i.e., the white noise—no correlation in time, with the power spectrum $S(f) \sim f^0$—and the integral of the white noise, the Brownian noise (Wiener process)—no correlation between increments, with the power spectrum $S(f) \sim f^{-2}$—are very well known and understood. $1/f^{\beta}$ noise with $0 < \beta < 2$, however, cannot be realized and explained in a similar manner and, therefore, no generally recognized explanation of the ubiquity of $1/f$ noise has yet been proposed.

Despite the numerous models and theories proposed since its discovery more than 80 years ago [1], the intrinsic origin of $1/f$ noise still remains an open question. Although in recent years about 100 papers have been published annually with the phrase '$1/f$ noise', '$1/f$ fluctuations' or 'flicker noise' in the title, there is no conventional picture of the phenomenon and the mechanisms leading to $1/f$ fluctuations are not often clear. Most of the models and theories have restricted validity because of the assumptions specific to the problem under consideration. Categorization and summarizing for the contemporary stage of theories and models of $1/f$ noise are rather problematic: on one hand, due to the abundance and variety of the proposed approaches, and on the other hand, because of the absence of a recent comprehensive review of the wide-ranging 'problem of $1/f$ noise' and because of the lack of a survey summarizing the current theories and models of $1/f$ noise. We can cite only a pedagogical review of the $1/f$ noise subject by Milotti [12]. Li presents a kind of classification by category of publication related to 1/f noise up to 2007 [13]. In the peer-reviewed encyclopedia Scholarpedia [14], there is also a short current review on the subject under the consideration. Thus, we present here only a short and partial categorization of $1/f$ noise models with a restricted list of references.

Until recently, probably the most general and common models, theories and explanations of $1/f$ noise have been based on some formal mathematical description such as fractional Brownian motion, the half-integral of the white noise, or some algorithms for generation of signals with scaled properties [15] and the popular modeling of $1/f$ noise as the superposition of independent elementary processes with the Lorentzian spectra and a proper distribution of relaxation times, e.g., a $1/\tau_{\mathrm{relax}}$ distribution [16]. The weakness of the latter approach is that the simulation of $1/f^{\beta}$ noise with the desirable slope β requires finding the special distributions of parameters of the system under consideration; at least a wide range of relaxation time constants should be assumed in order to allow correlation with experiments (see also [5, 6, 10, 17]).

Models of $1/f$ noise in some particular systems are usually specific and do not explain the omnipresence of processes with the $1/f^{\beta}$ spectrum. Predominantly, the $1/f$ noise problem has been analyzed for conducting media, semiconductors, metals, electronic devices and other electronic systems [1, 5, 6, 16, 18]. The topic of $1/f$ noise in such systems has been comprehensively reviewed [5], even recently [6]. Nevertheless, despite numerous suggested models, the origin of flicker noise even there still remains an open issue: 'more and more studies suggest that if there is a common regime for the low frequency noise, it must be mathematical rather than the physical one' [6]. Here we can additionally mention the disputed quantum theory [19] of $1/f$ noise and $1/f$ noise satisfactorily interpreted in quantum chaos [20].

In 1987 Bak *et al* [21, 22] introduced the notion of self-organized criticality (SOC) with one of the main motivations being that of explaining the universality of $1/f$ noise. SOC systems are non-equilibrium systems driven by their own dynamics to a self-organization. Fluctuations around this state, the so-called avalanches, are characterized by power-law distributions in time and space, criticality implying long-range correlations. The distributions of avalanche sizes, durations and energies are all seen to be power laws.

Two types of correlation should be distinguished in SOC: the scale-free distribution of their avalanche sizes and temporal correlations between avalanches, bursts or (rare, extreme) events. In the standard SOC models the search for $1/f^{\beta}$ noise is based on the observable power-law dependence of the burst size as a function of the burst duration and the power-law distribution of the burst sizes, with Poisson distributed interevent times. Such power laws usually result in relatively high frequency power law, $1/f^{\beta}$, behavior of the power spectrum with the exponent $1.4 \lesssim \beta \lesssim 2$ [23, 24]. This mechanism for the power-law spectrum is related to the statistical models of $1/f^{\beta}$ noise representing signals as consisting of different random pulses [25]–[27].

It should also be mentioned that originally SOC was suggested as an explanation of the occurrence of $1/f$ noise and fractal pattern formation in the dynamical evolution of certain systems. However, recent research has revealed that the connection between these and SOC is rather loose [28]. Though an explanation of $1/f$ noise was one of the main motivations for the initial proposal of SOC, time dependent properties of self-organized critical systems have not been studied much theoretically so far [29].

It is of interest to note that the paper [21] is the most cited paper in the field of $1/f$ noise problems, but it was shown later on [23, 24] that the mechanism proposed in [21] for SOC systems results in $1/f^{\beta}$ fluctuations with $1.5 \lesssim \beta \lesssim 2$ and does not explain the

J. Stat. Mech. (2009) P02051

omnipresence of $1/f$ noise. On the other hand, we can point to a recent paper [30] where an example of $1/f$ noise in the classical sandpile model has been provided.

It should be emphasized, however, that another mechanism of $1/f^{\beta}$ noise, based on the temporal correlations between avalanches, bursts or (rare, extreme) events, may be the source of the power-law $1/f^{\beta}$ spectra with $\beta \lesssim 1$ [31]. Moreover, SOC is closely related to the observable $1/f^{\beta}$ crackling noise [32], Barkhausen noise [33], fluctuations of the long-term correlated seismic events [34] and $1/f^{\beta}$ fluctuations for non-equilibrium phase transitions [35].

Ten years ago we proposed [8, 9], analyzed [36] and later on generalized [10] a simple point process model of $1/f^{\beta}$ noise and applied it to financial systems [37]. Moreover, starting from the point process model we derived stochastic nonlinear differential equations, i.e., general Langevin equations with a multiplicative noise for the signal intensity exhibiting $1/f^{\beta}$ noise (with $0.5 \leq \beta \leq 2$) in any desirable wide range of frequency f [38]. Here we analyze the scaling properties of the signals generated by the particular stochastic differential equations. We obtain and analyze the power-law dependences of the signal intensity, power spectrum, autocorrelation functions and the second-order structural functions. The comparison with the numerical simulations is presented.

Moreover, the numerical analysis reveals a second (recall that we start from the point process) structure of the signal composed of peaks, bursts, clusters of events with the power-law distributed burst sizes S, burst durations T and the interburst time θ, while the burst sizes are approximately proportional to the squared durations of the bursts, $S \sim T^2$. Therefore, the proposed nonlinear stochastic model may simulate SOC and other similar systems where the processes consist of avalanches, bursts or clustering of the extreme events [21]–[24], [28]–[35], [39].

2. The model

We start from the point process

$$x(t) = a \sum_k \delta(t - t_k), \tag{1}$$

representing the signal, current, flow, etc, $x(t)$, as a sequence of correlated pulses or series of events. Here $\delta(t)$ is the Dirac δ-function and a is a contribution to the signal $x(t)$ of one pulse at the time moment t_k. Our model is based on the generic multiplicative process for the interevent time $\tau_k = t_{k+1} - t_k$,

$$\tau_{k+1} = \tau_k + \gamma \tau_k^{2\mu - 1} + \sigma \tau_k^{\mu} \varepsilon_k, \tag{2}$$

generating the power-law distributed

$$P_k(\tau_k) \sim \tau_k^{\alpha}, \qquad \alpha = \frac{2\gamma}{\sigma^2} - 2\mu \tag{3}$$

sequence of the interevent times τ_k [10, 37].

Some motivations for equation (2) were given in papers [8, 10, 36, 37]. Additional comments are presented below, after equation (6).

Therefore, in our model the (average) interevent time τ_k fluctuates due to the random perturbations by a sequence of uncorrelated normally distributed random variables $\{\varepsilon_k\}$

J. Stat. Mech. (2009) P02051

with zero expectation and unit variance; σ is the standard deviation of the white noise and $\gamma \ll 1$ is a coefficient of the nonlinear damping.

Transition from the occurrence number k to the actual time t in equation (2) according to the relation $\mathrm{d}t = \tau_k \, \mathrm{d}k$ yields the Itô stochastic differential equation for the variable $\tau(t)$ as a function of the actual time,

$$\mathrm{d}\tau = \gamma \tau^{2\mu-2} \, \mathrm{d}t + \sigma \tau^{\mu-1/2} \, \mathrm{d}W, \tag{4}$$

where W is a standard Wiener process. Equation (4) generates the stochastic variable τ, power-law distributed,

$$P_t(\tau) = \frac{P_k(\tau)}{\langle \tau_k \rangle} \tau \sim \tau^{\alpha+1}, \tag{5}$$

in the actual time t. Here $\langle \tau_k \rangle$ is the average interevent time. $\tau(t)$ may be interpreted as the average time dependent interevent time of the modulated Poisson-like process with the distribution of the interevent time

$$P_p(\tau_p) = \frac{1}{\tau\,(t)} \mathrm{e}^{-\tau_p/\tau(t)} = n\,(t)\, \mathrm{e}^{-n(t)\tau_p}, \tag{6}$$

where $n(t) = 1/\tau(t)$ is the time dependent rate of the process [37].

Additional support for the stochastic model (1)–(6) of the scaled processes and $1/f^{\beta}$ noise is the following. The fluctuations of the intensity of the signals, currents, flows, etc, consisting of discrete objects (electrons, photons, packets, vehicles, pulses, events, etc), are primarily and basically defined in terms of fluctuations of the (average) interevent, interpulse, interarrival, recurrence, or waiting time. Equation (4) is a special case of the general nonlinear Langevin equation

$$\mathrm{d}\tau = d(\tau) \, \mathrm{d}t + b(\tau) \, \mathrm{d}W(t), \tag{7}$$

with the drift coefficient $d(\tau)$ and a multiplicative noise $b(\tau)\xi(t)$ for the (average) interevent time $\tau(t)$, with $\xi(t)$ being a white nose defined from the relation $\mathrm{d}W(t) = \xi(t)\,\mathrm{d}(t)$. Equation (7) is a straight analogy of the well-known Langevin equation for the continuous random variable x. For the process consisting of the discrete objects the intensity of the signal fluctuates due to fluctuations of the rate, i.e., density of the objects in the time axis, which is a consequence of fluctuations of the interarrival or interevent time. Equation (7) in reality represents (in the simplest form) such fluctuations due to random perturbations by white noise.

In papers [8]–[10] it has been shown that the small interevent times and clustering of the events make the greatest contribution to $1/f^{\beta}$ noise, low frequency fluctuations and exhibiting of long-range scaled features. Therefore, it is straightforward to approximate the nonlinear diffusion coefficient $b(\tau)$ and the distribution of the interevent time $P_t(\tau)$ in some interval of small interevent times τ by the power-law dependences or expansions

$$b(\tau) = \sigma \tau^{\mu-(1/2)}, \tag{8}$$

$$P_t(\tau) \sim \tau^{\alpha+1}. \tag{9}$$

The power-law distribution of the interevent, recurrence, or waiting time is observable in different systems from physics and seismology to the Internet, financial markets

J. Stat. Mech. (2009) P02051

and neural spikes (see, e.g., [10, 37, 39]). It should be noted that the multiplicative equations with the drift coefficient $d(\tau)$ proportional to the Stratonovich correction for the drift, leading the transformation from the Stratonovich to the Itô stochastic differential equation [40], i.e., when

$$d(\tau) \sim \tfrac{1}{2} b(\tau) b'(\tau), \tag{10}$$

with the power-law dependent, like (8), diffusion coefficient $b(\tau)$, generate the power-law distribution of the stochastic variable. Equations (2) and (4) are definitely of such a kind. Therefore, equation (4) is one of the simplest multiplicative equations for the interevent time, modeling scaled processes, while equation (2) is just the lowest order difference equation following from equation (4) when the step of integration Δt_k equals the interevent time τ_k.

The Itô transformation in equation (4) of the variable from τ to the intensity averaged over the time interval τ of the signal $x(t) = a/\tau(t)$ [38] yields the class of Itô stochastic differential equations

$$\mathrm{d}x = \left(\eta - \tfrac{1}{2}\lambda\right) x^{2\eta-1}\, \mathrm{d}t_{\mathrm{s}} + x^{\eta}\, \mathrm{d}W \tag{11}$$

for the signal as a function of the scaled time

$$t_{\mathrm{s}} = \frac{\sigma^2}{a^{3-2\mu}} t. \tag{12}$$

Here the new parameters

$$\eta = \tfrac{5}{2} - \mu, \qquad \lambda = 3 + \alpha = \frac{2\gamma}{\sigma^2} + 2(\eta - 1) \tag{13}$$

have been introduced.

The Fokker–Planck equation associated with equation (11) gives the power-law distribution density of the signal intensity

$$P(x) \sim \frac{1}{x^{\lambda}}, \tag{14}$$

with the exponent λ.

For $\lambda > 1$, distribution (14) diverges as $x \to 0$, and, therefore, the diffusion of x should be restricted at least from the side of small values, or equation (11) should be modified. Thus, further on, we will consider the modified equation for $x > 0$ only,

$$\mathrm{d}x = \left(\eta - \tfrac{1}{2}\lambda\right) (x_m + x)^{2\eta-1}\, \mathrm{d}t_{\mathrm{s}} + (x_m + x)^{\eta}\, \mathrm{d}W, \tag{15}$$

with the additional small parameter x_m restricting the divergence of the power-law distribution of x at $x = 0$.

Equation (15) for small $x \ll x_m$ represents the linear additive stochastic process generating the Brownian motion with the steady drift, while for $x \gg x_m$ it reduces to the multiplicative equation (11).

3. Analysis of the model

The Fokker–Planck equation associated with equation (15) gives the steady-state solution for the distribution of x,

$$P(x) = \frac{(\lambda-1)x_m^{\lambda-1}}{(x_m+x)^{\lambda}}, \qquad x>0, \quad \lambda>1. \tag{16}$$

We can obtain the power spectral density of the signal generated by equation (15) from equation (28) derived in paper [10]. After some algebra we can write

$$S(f) = \frac{A}{f^{\beta}}, \qquad f \gg f_1 = \frac{2+\lambda-2\eta}{2\pi} x_m^{2(\eta-1)}, \tag{17}$$

with

$$A = \frac{(\lambda-1)\Gamma(\beta-1/2)x_m^{\lambda-1}}{2\sqrt{\pi}(\eta-1)\sin(\pi\beta/2)}\left(\frac{2+\lambda-2\eta}{2\pi}\right)^{\beta-1} \tag{18}$$

and

$$\beta = 1 + \frac{\lambda-3}{2(\eta-1)} \tag{19}$$

for $0.5<\beta<2$, $4-\eta<\lambda<1+2\eta$ and $\eta>1$. Note that the frequency f in equation (17) is the scaled frequency matching the scaled time t_s (12).

The autocorrelation function $C(s)$ of the process can be expressed according to Wiener–Khinchin theorem as the inverse Fourier transform of the power spectrum,

$$C(s) = \langle x(t)x(t+s)\rangle = \int_0^{\infty} S(f)\cos(2\pi f s)\,\mathrm{d}f. \tag{20}$$

A pure $1/f^{\beta}$ power spectrum is physically impossible because the total power would be infinity. Depending on whether β is greater or less than 1, it is necessary to introduce a low frequency cutoff $f_{\min}$ or a high frequency cutoff $f_{\max}$ [41, 42]. For calculation of the autocorrelation function according to equation (20), when $\beta>0$ it is not necessary to introduce the high frequency cutoff.

Usually one introduces a discontinuous transition to the flat spectrum at the lower cutoff $f_{\min}$ [41, 42]. Here at low frequencies we will insert the smooth transition to the flat spectrum in the vicinity of $f_0 \sim f_{\min}$, i.e., we will approximate the power spectrum (17) as

$$S(f) = \frac{A}{(f_0^2+f^2)^{\beta/2}} = \begin{cases} A/f_0^{\beta}, & f \ll f_0, \\ A/f^{\beta}, & f \gg f_0. \end{cases} \tag{21}$$

Inserting (21) into equation (20) we obtain

$$C(s) = \frac{A\sqrt{\pi}}{\Gamma(\beta/2)f_0^{\beta-1}}\left(\frac{z}{2}\right)^{h} K_h(z), \tag{22}$$

where $K_h(z)$ is the modified Bessel function, $z = 2\pi f_0 s$ and $h = (\beta-1)/2$.

J. Stat. Mech. (2009) P02051

The first two terms of the expansion of equation (22) in powers of z are

$$C(s) = \frac{A\sqrt{\pi}}{2\Gamma(\beta/2) f_0^{\beta-1}} \left[\Gamma\left(\frac{\beta-1}{2}\right) + \Gamma\left(\frac{1-\beta}{2}\right) (\pi f_0 s)^{\beta-1} \right] \tag{23}$$

for $h \neq 0$, i.e., $\beta \neq 1$, and

$$C(s) = AK_0(2\pi f_0 s) \simeq A[-\gamma - \ln(\pi f_0 s)] = \mathrm{const} - A \ln s \tag{24}$$

for $h = 0$, i.e., for the pure $1/f$ noise with $\beta = 1$. Here $\gamma \simeq 0.577\,216$ is Euler's constant.

The leading terms of expression (23) differ depending on whether $\beta < 1$ or $\beta > 1$. Thus for $h < 0$, i.e., when $0 < \beta < 1$,

$$C(s) = \frac{A\Gamma(1-\beta)}{(2\pi s)^{1-\beta}} \sin\left(\frac{\pi\beta}{2}\right) \sim \frac{1}{s^{1-\beta}}, \tag{25}$$

while for $h > 0$, i.e., for $1 < \beta < 3$,

$$C(s) = C(0) - Bs^{\beta-1}. \tag{26}$$

Here

$$C(0) = \langle x^2 \rangle = \int_0^{\infty} S(f)\, \mathrm{d}f = \frac{A\sqrt{\pi}\Gamma\left((\beta-1)/2\right)}{2 f_0^{\beta-1} \Gamma\left(\beta/2\right)} \tag{27}$$

and

$$B = \frac{A\pi^{\beta+(1/2)}}{2\Gamma\left(\beta/2\right)\Gamma\left((\beta+1)/2\right)\sin\left(\pi(\beta-1)/2\right)} = -A(2\pi)^{\beta-1}\Gamma(1-\beta)\sin\left(\frac{\pi\beta}{2}\right). \tag{28}$$

For $\beta = 2$ equations (22) and (26)–(28) yield

$$C(s) = C(0)\mathrm{e}^{-2\pi f_0 s} = \frac{A\pi}{2f_0}\mathrm{e}^{-2\pi f_0 s} = C(0) - A\pi^2 s \pm \cdots. \tag{29}$$

It should be noted that particular cases (24)–(29) of the general expressions (22) and (23) are in agreement with the results of papers [41]–[43] obtained with the non-uniform cutoff of the spectrum at low frequency. On the other hand, the parameter h introduced for $\beta \geq 1$ coincides with the Hurst exponent H [42],

$$H = \begin{cases} 0, & 0 < \beta \leq 1, \\ \frac{1}{2}(\beta-1), & 1 < \beta < 3, \\ 1, & 3 \leq \beta < 4. \end{cases} \tag{30}$$

The exponent H is associated with the scaling of the second-order structural function, or height–height correlation function [41]–[44],

$$F(s) = F_2^2(s) = \langle |x(t+s) - x(t)|^2 \rangle \sim s^{2H}. \tag{31}$$

The exponent H characterizes the power-law diffusion rate, as well. This variance of the differenced time series (delayed signal) may be expressed as

$$F(s) = \langle x^2(t+s)\rangle + \langle x^2(t)\rangle - 2\langle x(t+s)x(t)\rangle = 2[C(0) - C(s)]. \tag{32}$$

J. Stat. Mech. (2009) P02051

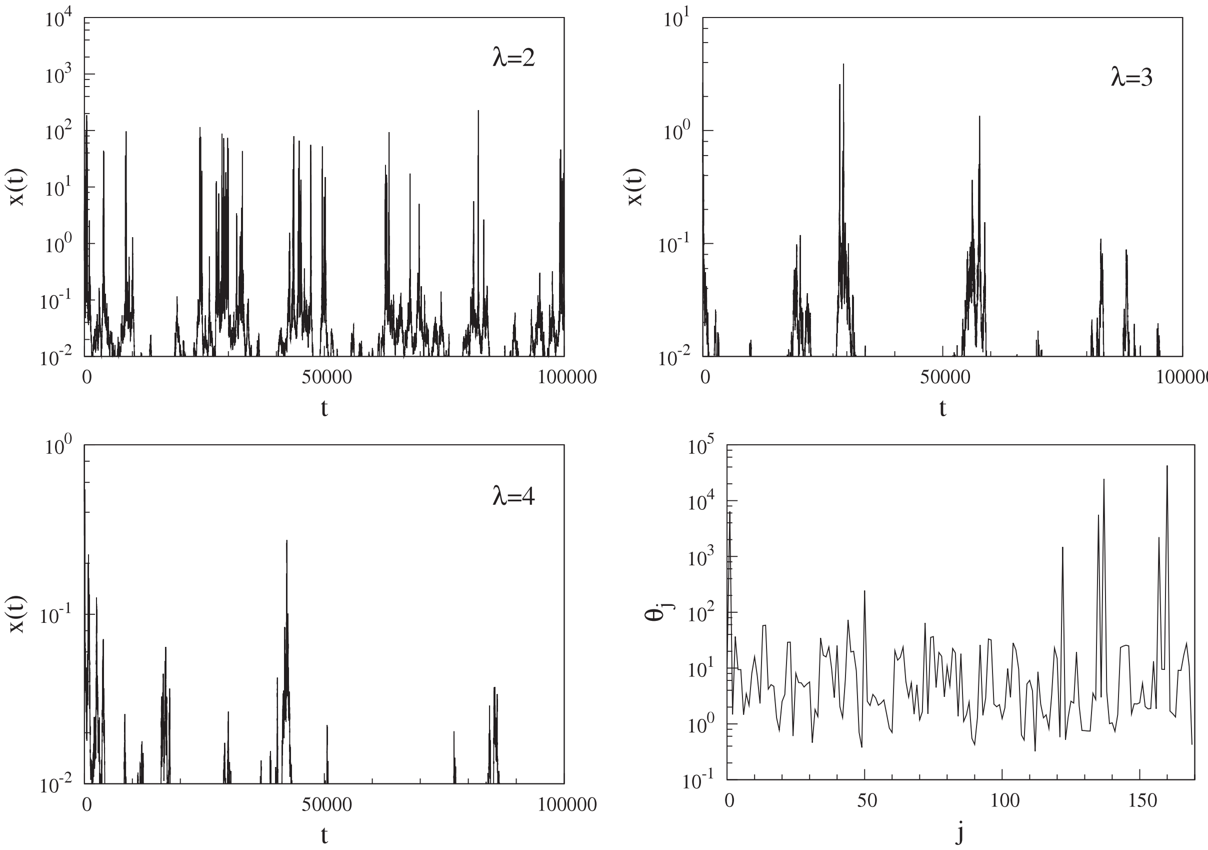

Figure 1. Examples of the numerically computed signals according to equations (37) and (38) with the parameters $\eta = 2$, $x_m = 10^{-2}$ and different values of β and λ: $\beta = 1/2$ when $\lambda = 2$, $\beta = 1$ when $\lambda = 3$ and $\beta = 3/2$ when $\lambda = 4$, and the interburst time θ_j as a function of the occurrence number j of the events peaking above the threshold value $x_{\mathrm{th}} = 0.1$ for the pure $1/f$ noise with $\beta = 1$ when $\lambda = 3$.

Substituting expressions (20) and (27) into (32) we have

$$F(s) = F_2^2(s) = 4\int_0^\infty S(f)\sin^2(\pi s f)\,\mathrm{d}f. \tag{33}$$

For the convergence of the integral in (33) at $\beta \le 1$ we need to cut off the power-law spectrum (17) at high frequency $f_{\max}$. Then the leading terms of the height–height correlation function (33) are

$$F(s) = 2A \times \begin{cases} \dfrac{f_{\max}^{1-\beta}}{1-\beta}\left[1 - \dfrac{\Gamma(2-\beta)}{(2\pi f_{\max}s)^{1-\beta}}\sin\left(\dfrac{\pi\beta}{2}\right)\right], & 0 < \beta < 1, \\ [\ln(\pi f_{\max}s) - \gamma], & \beta = 1. \end{cases} \tag{34}$$

For $1 < \beta < 3$ the integral in (33) may be integrated exactly and we have

$$F(s) = -2A\Gamma(1-\beta)\sin\left(\frac{\pi\beta}{2}\right)(2\pi s)^{\beta-1} = 2Bs^{\beta-1}, \qquad 1 < \beta < 3. \tag{35}$$

J. Stat. Mech. (2009) P02051

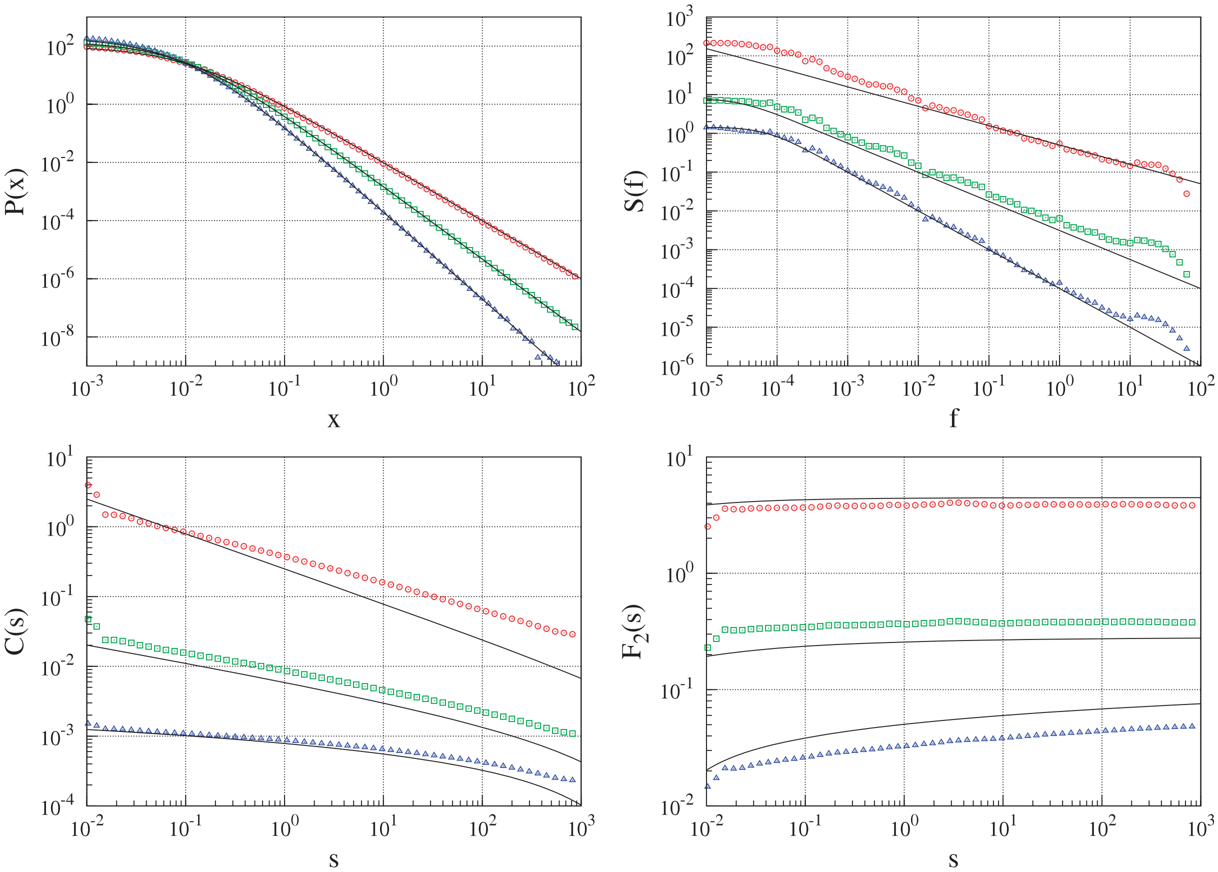

Figure 2. Distribution density, $P(x)$, power spectral density, $S(f)$, autocorrelation function, $C(s)$, and the second-order structural function, $F_2(s)$, for solutions of equation (15) with $\eta = 2$, $x_m = 10^{-2}$ and different values of the parameter λ: $\lambda = 2$ (circles), $\lambda = 2.5$ (squares) and $\lambda = 3$ (triangles) in comparison with the analytical results (solid lines) according to equations (16), (18)–(22), (34) and (35), respectively.

The spatial power spectrum and the height–height correlation function (31) are used for analysis of rough self-affine surfaces and assessing the growth mechanism of thin films [45]–[49], as well. There sometimes the violation of the scaling relation $\beta = 2H + 1$ is observable [46, 47, 50].

4. Numerical analysis

For the numerical analysis we have to solve equation (15) and analyze the numerical solutions obtained. We can solve equation (15) using the method of discretization with the variable step of integration

$$h_i = \Delta t_i = \frac{\kappa^2}{(x_m + x_i)^l}, \tag{36}$$

where κ is a small parameter while the exponent l rules the dependence of the integration step on the value of the variable x. Thus, $l = 0$ corresponds to the fixed step, for $l = 1$

J. Stat. Mech. (2009) P02051

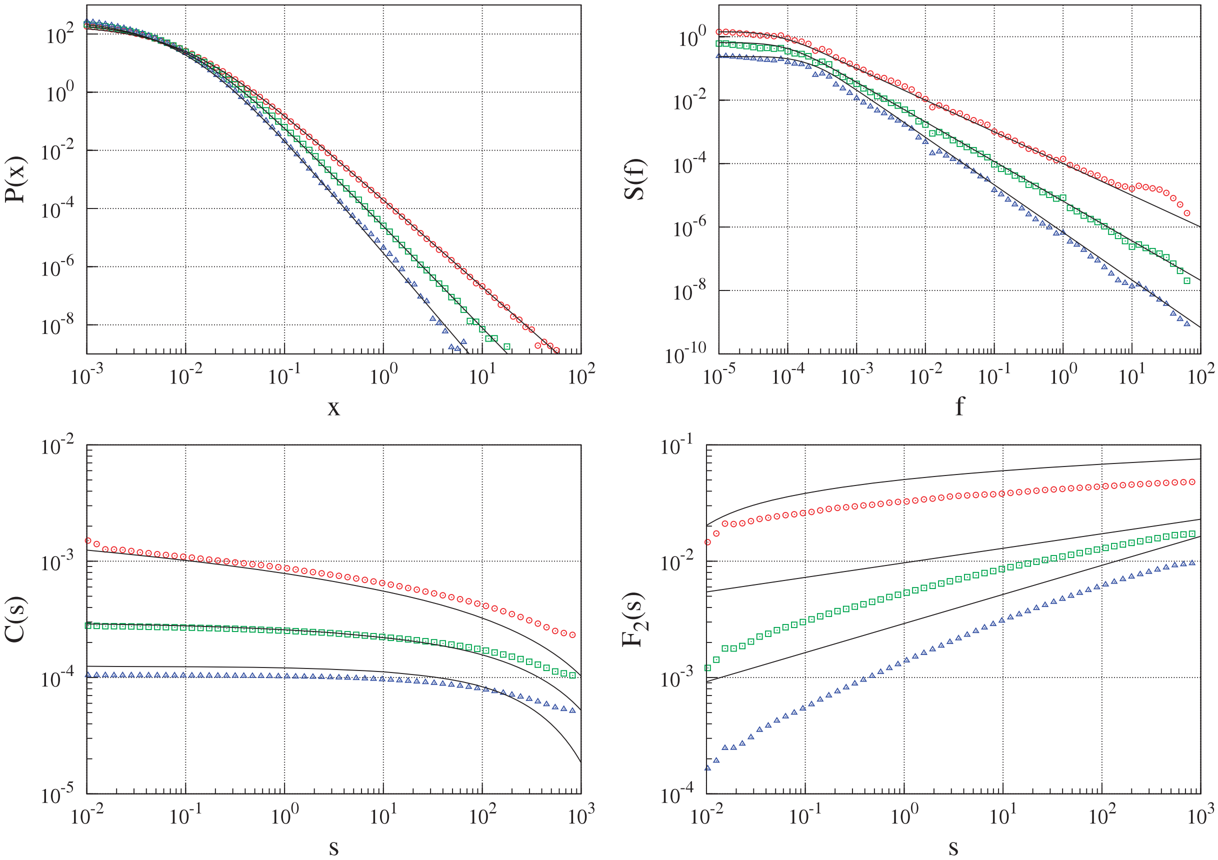

Figure 3. As figure 2 but for the parameters $\lambda = 3$ (circles), $\lambda = 3.5$ (squares) and $\lambda = 4$ (triangles).

we have an analogy with equation (2) when the step is proportional to the interevent time τ_k, $l = 2(\eta - 1)$ matches the case when the change of the variable x in one step is proportional to the value of the variable at the time of the step [38] and so on. As a result we have the system of the difference equations

$$x_{i+1} = x_i + \kappa^2 \left(\eta - \tfrac{1}{2}\lambda\right) (x_m + x_i)^{2\eta-1-l} + \kappa (x_m + x_i)^{\eta-l/2} \varepsilon_i, \tag{37}$$

$$t_{i+1} = t_i + \frac{\kappa^2}{(x_m + x_i)^l}, \qquad x_i > 0. \tag{38}$$

Here ε_i is a set of uncorrelated normally distributed random variables with zero expectation and unit variance. In the Milstein approximation, equation (37) should be replaced by the equation

$$x_{i+1} = x_i + \frac{\kappa^2}{2} (\eta - \lambda) (x_m + x_i)^{2\eta-1-l} + \kappa ({}_m + x_i)^{\eta-l/2} \varepsilon_i + \frac{\kappa^2 \eta}{2} (x_m + x_i)^{2\eta-1-l} \varepsilon_i^2. \tag{39}$$

Numerical analysis indicates that the variable of equation (15) exhibits some peaks, bursts or extreme events, corresponding to the large deviations of the variable from the appropriate average value. As examples, in figure 1 we show the illustrations of the signals generated according to equation (15) for different slopes of the signal distributions and

J. Stat. Mech. (2009) P02051

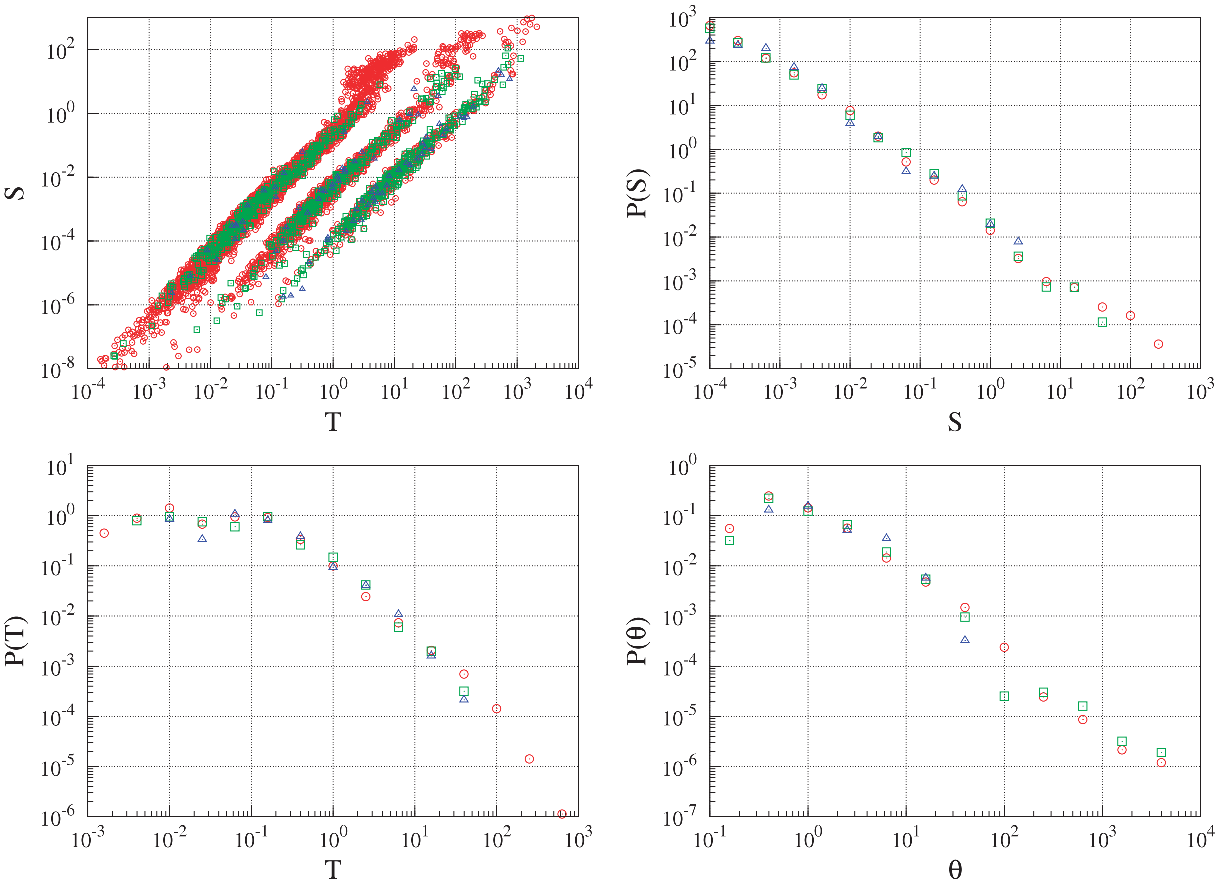

Figure 4. Dependence of the burst size S as a function of the burst duration T: traces from the top of the figure for size S above the threshold value $x_{\rm th} = 0.02$, 0.1 and 0.5, respectively, distributions of the burst size, $P(S)$, burst duration, $P(T)$, and interburst time, $P(\theta)$, for the peaks above the threshold value $x_{\rm th} = 0.1$. Calculations are as in figures 2 and 3 with the parameters $\eta = 2$, $x_m = 10^{-2}$ and different values of the parameter λ: $\lambda = 2$ (circles), $\lambda = 3$ (squares) and $\lambda = 4$ (triangles).

the dependence of the interburst time θ_j on the burst occurrence number j. We see that the computed signal is composed of bursts of different sizes with a wide-range distribution of the interburst time. In figures 2 and 3 the numerical calculations of the distribution density, $P(x)$, power spectral density, $S(f)$, autocorrelation function, $C(s)$, and second-order structural function, $F_2(s) = \sqrt{F(s)}$, for solutions of equation (15) with $\eta = 2$, $x_m = 10^{-2}$ and different values of the parameter λ, are presented. We see rather good agreement between the numerical calculations and the analytical results except for the structural function $F_2(s)$ when $\lambda > 3$. Numerical evaluation of the structural function in the case of a steep power-law distribution is problematic, because in the calculation one needs to average (squared) small differences of the rare large fluctuations.

In figure 4 we demonstrate numerically that the size of the generated bursts S is approximately proportional to the squared burst duration T, i.e., $S \propto T^2$, and asymptotically approximately power-law distributions of the burst size, $P(S) \sim S^{-1.3}$,

J. Stat. Mech. (2009) P02051

burst duration, $P(T) \sim T^{-1.5}$, and interburst time, $P(\theta) \sim \theta^{-1.5}$, for the peaks above the threshold value x_{th} of the variable $x(t)$.

It should be noted that the parameter $\eta = 2$ yields in equation (4) the additive noise and the linear relaxation of the signal $x = a/\tau$, i.e., the simple (pure) Brownian motion in the actual time of the interevent time with the linear relaxation of the signal.

5. Conclusions

Starting from the multiplicative point process we obtain stochastic nonlinear differential equations which generate signals with power-law statistics, including $1/f^{\beta}$ fluctuations. We derive analytical expressions for the probability density of the signal, the power spectral density, the autocorrelation function, and the second-order structural function and demonstrate that the analytical results are in agreement with the results of numerical simulations. The numerical analysis of the equations reveals the secondary structure of the signal composed of peaks or bursts, corresponding to the large deviations of the variable x from the proper average fluctuations. The burst sizes are approximately proportional to the squared duration of the burst. According to the theory [24, 27] such dependence for the uncorrelated bursts should result in $1/f^{\beta}$ noise with $\beta \approx 2$ in the relatively high frequency region. The power-law distribution $P(\theta)$ of the interburst time θ indicates correlation of the burst occurrence times and may result in $1/f^{\beta}$ noise with $\beta < 2$, similarly to the point process model [8]–[10]. On the other hand, the proposed model not only reproduces $1/f$ noise models and the processes in SOC and crackling systems, but also is related to the clustering Poisson process [51], $1/f$ noise due to diffusion of defects or impurity centers in semiconductors [52], $1/f$ noise in nanochannels, single-channel and ion channel currents [53], etc. Therefore, the model presented and analyzed may be used for simulation of the long-range scaled processes exhibiting $1/f$ noise, power-law distributions and self-organization.

References

[1] Johnson J B, 1925 *Phys. Rev.* **26** 71
Schottky W, 1926 *Phys. Rev.* **28** 74
[2] Mandelbrot B B, 1963 *J. Bus.* **36** 394
Allan D W, 1966 *Proc. IEEE* **54** 221
Granger C W J, 1966 *Econometrica* **34** 150
[3] Beran J, 1994 *Statistics for Long-Memory Processes* (New York: Chapman and Hall)
Mandelbrot B B, 1997 *Fractals and Scaling in Finance* (New York: Springer)
Mandelbrot B B, 1999 *Multifractals and 1/f Noise: Wild Self-Affinity in Physics* (New York: Springer)
Mantegna R N and Stanley H E, 2000 *An Introduction to Econophysics: Correlations and Complexity in Finance* (Cambridge: Cambridge University Press)
Racher S T (ed), 2003 *Handbook of Heavy Tailed Distributions in Finance* (Amsterdam: Elsevier)
[4] Voss R F and Clarke J, 1975 *Nature* **258** 317
Press V H, 1978 *Comments Astrophys.* **7** 103
Gardner M, 1978 *Sci. Am.* **238** 16
Newman M E J, 2005 *Contemp. Phys.* **46** 323
[5] Hooge F N, Kleinpenning T G M and Vadamme L K J, 1981 *Rep. Prog. Phys.* **44** 479
Dutta P and Horn P M, 1981 *Rev. Mod. Phys.* **53** 497
Kogan S M, 1985 *Usp. Fiz. Nauk.* **145** 285
Kogan S M, 1985 *Sov. Phys. Usp.* **28** 170 (translation)
Weissman M B, 1988 *Rev. Mod. Phys.* **60** 537
Van Vliet C M, 1991 *Solid-State Electron.* **34** 1

Zhigalskii G P, 1997 *Usp. Fiz. Nauk.* **167** 623
Zhigalskii G P, 1997 *Sov. Phys. Usp.* **40** 599 (translation)
Zhigal'skii G P, 2003 *Usp. Fiz. Nauk.* **56** 449
[6] Wong H, 2003 *Microelectron. Reliab.* **43** 585
[7] Musha T and Higuchi H, 1976 *Japan. J. Appl. Phys.* **15** 1271
Lawrence A, Watson M G, Pounds K A and Elvis M, 1987 *Nature* **325** 695
Gilden D L, Thornton T and Mallon M W, 1995 *Science* **267** 1837
Gabaix X, Gopikrishnan P, Plerou V and Stanley E, 2003 *Nature* **423** 267
Carbone A, Kaniadakis G and Scarfone A M, 2007 *Eur. Phys. J.* B **57** 121
Smeets R M M, Keyser U F, Dekker N H and Dekker C, 2008 *Proc. Nat. Acad. Sci.* **105** 417
[8] Kaulakys B and Meškauskas T, 1998 *Phys. Rev.* E **58** 7013
[9] Kaulakys B, 1999 *Phys. Lett.* A **257** 37
[10] Kaulakys B, Gontis V and Alaburda M, 2005 *Phys. Rev.* E **71** 051105
[11] Li W, 2009 http://www.nslij-genetics.org/wli/1fnoise
[12] Milotti E, 2002 arXiv:physics/0204033
[13] Li W, 2007 http://www.nslij-genetics.org/wli/1fnoise/index-by-category.html
[14] Scholarpedia 2009 http://www.scholarpedia.org/article/1/f_noise
[15] Mandelbrot B B and Van Ness J W, 1968 *SIAM Rev.* **10** 422
Masry E, 1991 *IEEE Trans. Inform. Theory* **37** 1173
Milotti E, 1995 *Phys. Rev.* E **51** 3087
Mamontov Y V and Willander M, 1997 *Nonlinear Dyn.* **12** 399
Ninness B, 1998 *IEEE Trans. Inform. Theory* **44** 32
Jumarie G, 2005 *Appl. Math. Lett.* **18** 739
Milotti E, 2005 *Phys. Rev.* E **72** 056701
Ostry D I, 2006 *IEEE Trans. Inform. Theory* **52** 1609
Erland S and Greenwood P E, 2007 *Phys. Rev.* E **76** 031114
[16] Bernamont J, 1937 *Ann. Phys.* **7** 71
Surdin M, 1939 *J. Phys. Radium* **10** 188
Van der Ziel A, 1950 *Physica (Amsterdam)* **16** 359
McWhorter A L, 1957 *Semiconductor Surface Physics* ed R H Kingston (Philadelphia, PA: University of Pennsylvania) p 207
Ralls K S, Skocpol W J, Jackel L D, Howard R E, Fetter L A, Epworth R W and Tennant D M, 1984 *Phys. Rev. Lett.* **52** 228
Rogers C T and Buhrman R A, 1984 *Phys. Rev. Lett.* **53** 1272
Palenskis V, 1990 *Litov. Fiz. Sb.* **30** 107
Palenskis V, 1990 *Lith. Phys. J.* **30** (2) 1
Watanabe S, 2005 *J. Korean Phys. Soc.* **46** 646
[17] Kaulakys B, Alaburda M and Ruseckas J, 2007 *AIP Conf. Proc.* **922** 439
[18] Palenskis V and Shoblitskas Z, 1982 *Solid State Commun.* **43** 761
Kogan S M and Nagaev K E, 1984 *Solid State Commun.* **49** 387
Shtengel K and Yu C C, 2003 *Phys. Rev.* B **67** 165106
Burin A L, Shklowskii B I, Kozub V I, Galperin Y M and Vinokur V, 2006 *Phys. Rev.* B **74** 075205
Muller J, von Molnar S, Ohno Y and Ohno H, 2006 *Phys. Rev. Lett.* **96** 186601
Koch R H, DiVincento D P and Clarke J, 2007 *Phys. Rev. Lett.* **98** 267003
de Sousa R, 2007 *Phys. Rev.* B **76** 245306
Burin A L, Shklowskii B I, Kozub V I, Galperin Y M and Vinokur V, 2008 *Phys. Status Solidi* c **5** 800
D'Souza A I, Stapelbroek M G, Robinson E W, Yoneyama C, Mills H A, Kinch M A, Skokan M R and Shih H D, 2008 *J. Electron. Mater.* **37** 1318
[19] Handel P H, 1975 *Phys. Rev. Lett.* **34** 1492
Handel P H, 1975 *Phys. Rev. Lett.* **34** 1495
Kazakov K A, 2008 *Physica* B **403** 2255
Foster S, 2008 *Phys. Rev.* A **78** 013820
[20] Relano A, Gomez J M G, Molina R A, Retamosa J and Faleiro E, 2002 *Phys. Rev. Lett.* **98** 244102
Salasnich L, 2005 *Phys. Rev.* E **71** 047202
Abul-Magd A Y, Dietz B, Friedrich T and Richter A, 2008 *Phys. Rev.* E **77** 046202
[21] Bak P, Tang C and Wiesenfeld K, 1987 *Phys. Rev. Lett.* **59** 381
[22] Bak P, Tang C and Wiesenfeld K, 1988 *Phys. Rev.* A **38** 364
[23] Jensen H J, Christensen K and Fogedby H C, 1989 *Phys. Rev.* B **40** 7425

Takayasu H, 1989 *Phys. Rev. Lett.* **63** 2563
Kertesz J and Kiss L B, 1990 *J. Phys. A: Math. Gen.* **23** L433
Jaeger H M and Nagel S R, 1992 *Science* **255** 1523
Paczuski M, Maslov S and Bak P, 1996 *Phys. Rev.* E **53** 414
Davidsen J and Schuster H G, 2000 *Phys. Rev.* E **62** 6111
Laurson L, Alava M J and Zapperi S, 2005 *J. Stat. Mech.* L11001
Laurson L and Alava M J, 2006 *Phys. Rev.* E **74** 066106
Poil S S, van Ooyen A and Linkenkaer-Hansen K, 2008 *Hum. Brain Mapping* **29** 770
[24] Kuntz M C and Sethna J P, 2000 *Phys. Rev.* B **62** 11699
[25] Heiden C, 1969 *Phys. Rev.* **188** 319
[26] Schick K L and Verveen A A, 1974 *Nature* **251** 599
[27] Ruseckas J, Kaulakys B and Alaburda M, 2003 *Lith. J. Phys.* **43** 223 [arXiv:0812.4674]
Kaulakys B, Alaburda M, Gontis V, Meškauskas T and Ruseckas J, 2007 *Traffic and Granular Flow'05* ed A Schadschneider, T Poschel, R Kuhne, M Schreckenberg and D E Wolf (Berlin: Springer) p 603 [arXiv:physics/0512068]
[28] Frigg R, 2003 *Stud. Hist. Phil. Sci.* **34** 613
[29] Dhar D, 2006 *Physica* A **369** 29
[30] Baiesi M and Maes C, 2006 *Europhys. Lett.* **75** 413
[31] Davidsen J and Paczuski M, 2002 *Phys. Rev.* E **66** 050101(R)
Bunde A, Eichner J F, Havlin S and Kantelhardt J W, 2004 *Physica* A **342** 308
Bunde A, Eichner J F, Kantelhardt J W and Havlin S, 2005 *Phys. Rev. Lett.* **94** 048701
Eichner J F, Kantelhardt J W, Bunde A and Havlin S, 2007 *Phys. Rev.* E **75** 011128
Blender R, Fraedrich K and Sienz F, 2008 *Nonlin. Processes Geophys.* **15** 557
[32] Sethna J P, Dahmen K A and Myers C R, 2001 *Nature* **410** 242
Eggenhoffner R, Celasco E and Celasco M, 2007 *Fluct. Noise Lett.* **7** L351
[33] Cote P J and Meisel L V, 1991 *Phys. Rev. Lett.* **67** 1334
Meisel L V and Cote P J, 1992 *Phys. Rev.* B **46** 10822
Colaiori F, 2008 *Adv. Phys.* **57** 287
[34] Telesca L, Cuomo V, Lapenna V and Macchiato M, 2002 *Fluct. Noise Lett.* **2** L357
Davidsen J, Grassberger P and Paczuski M, 2008 *Phys. Rev.* E **77** 066104
[35] Novak U and Usadel K D, 1991 *Phys. Rev.* B **43** 851
Koverda V P, Skokov V N and Skripov V P, 1998 *J. Exp. Theor. Phys.* **86** 953
Fukuda K, Takayasu H and Takayasu M, 2000 *Physica* A **287** 289
Koverda V P and Skokov V N, 2005 *Physica* A **346** 203
[36] Kaulakys B and Meškauskas T, 1999 *Nonlinear Anal.: Model. Control* **4** 87
Kaulakys B and Meškauskas T, 2000 *Microelecron. Reliab.* **40** 1781
Kaulakys B, 2000 *Microelecron. Reliab.* **40** 1787
Kaulakys B, 2000 *Lith. J. Phys.* **40** 281
[37] Gontis V and Kaulakys B, 2004 *Physica* A **343** 505
Gontis V and Kaulakys B, 2004 *Physica* A **344** 128
Gontis V and Kaulakys B, 2006 *J. Stat. Mech.* P10016
Gontis V and Kaulakys B, 2007 *Physica* A **382** 114
Gontis V, Kaulakys B and Ruseckas J, 2008 *Physica* A **387** 3891
[38] Kaulakys B and Ruseckas J, 2004 *Phys. Rev.* E **70** 020101(R)
Kaulakys B, Ruseckas J, Gontis V and Alaburda M, 2006 *Physica* A **365** 217
[39] Thurner S, Lowen S B, Feurstein M C, Heneghan C, Feichtinger H G and Teich M C, 1997 *Fractals* **5** 565
Schwalger T and Schimansky-Geier L, 2008 *Phys. Rev.* E **77** 031914
Perello J, Masoliver J, Kasprzak A and Kutner R, 2008 *Phys. Rev.* E **78** 036108
[40] Arnold P, 2000 *Phys. Rev.* E **61** 6091
Kupferman R, Pavliotis G A and Stuart A M, 2004 *Phys. Rev.* E **70** 036120
[41] Theiler J, 1991 *Phys. Lett.* A **155** 480
[42] Talocia S G, 1995 *Phys. Lett.* A **200** 264
[43] Caprari R S, 1998 *J. Phys. A: Math. Gen.* **31** 3205
[44] Kaulakys B, Alaburda M, Gontis V and Meškauskas T, 2006 *Conplexus Mundi: Emergent Patterns in Nature* ed M M Novak (Singapore: World Scientific) p 277

J. Stat. Mech. (2009) P02051

[45] Barabasi A L and Stanley H E, 1995 *Fractal Concepts in Surface Growth* (Cambridge: Cambridge University Press)
[46] Yang H-N and Lu T-M, 1995 *Phys. Rev.* B **51** 2479
[47] Zhao Y-P, Cheng C-F, Wang G-C and Lu T-M, 1998 *Surface Science* **409** L703
[48] Yanguas-Gil A, Cotrino J, Walkiewicz-Pietrzykowska A and Gonzalez-Elipe A R, 2007 *Phys. Rev.* B **76** 075314
[49] Palasantzas G, 2008 *J. Appl. Phys.* **104** 053524
[50] Timashev S F and Polyakov Y S, 2007 *Fluct. Noise Lett.* **7** R15
[51] Gruneis F, 1984 *Physica* A **123** 149
Gruneis F and Musha T, 1986 *Japan. J. Appl. Phys.* **25** 1504
Gruneis F, 2001 *Fluct. Noise Lett.* **1** R119
Takayasu H and Takayasu M, 2003 *Physica* A **324** 101
Gontis V, Kaulakys B and Ruseckas J, 2005 *AIP Conf. Proc.* **776** 144
[52] Gruneis F, 2000 *Physica* A **282** 108
Gruneis F, 2000 *Physica* A **290** 512 (Corrigendum)
Gruneis F, 2007 *Fluct. Noise Lett.* **7** C1
[53] Derksen H E and Verveen A A, 1966 *Science* **151** 1388
Bezrukov S M and Winterhalter M, 2000 *Phys. Rev. Lett.* **85** 202
Siwy Z and Fulinski A, 2002 *Phys. Rev. Lett.* **89** 158101
Banerjee J, Verma M K, Manna S and Ghosh S, 2006 *Europhys. Lett.* **73** 457
Kosinska I D and Fulinski A, 2008 *Europhys. Lett.* **81** 50006

J. Stat. Mech. (2009) P02051