Models for generation $1/f$ noise

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Abstract

Simple analytically solvable models are proposed exhibiting $1/f$ spectrum in a wide range of frequency. The signals of the models consist of pulses (point process), interevent times of which fluctuate about some average value, obeying an autoregressive process with very small damping. The power spectrum of the process can be expressed by the Hooge formula. The proposed models reveal possible origin of $1/f$ noise, i.e., random increments of the time intervals between pulses or interevent time of the process (Brownian motion in the time axis). © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

Widespread occurrence of signals exhibiting power spectral density with $1/f$ behavior suggests that a general mathematical explanation of such an effect might exist. However, physical models of $1/f$ noise in some physical systems are usually very specialized and they do not explain the omnipresence of the processes with $1/f^{\delta}$ spectrum. Mathematical algorithms and models for the generation of the processes with $1/f$ noise cannot, as a rule, be solved analytically and they do not reveal the origin as well as the necessary and sufficient conditions for the appearance of $1/f$ type fluctuations.

History of the progress in different areas of physics indicates the crucial influence of simple models on the advancement of the understanding of the main points of the new phenomena. We note here only the decisive influence of the Bohr model of hydrogen atom on the development of the quantum theory, the role of the Lorentz model as well as the logistic and standard (Chirikov) maps for an understanding of the deterministic chaos and the quantum kicked rotator for revealing the quantum localization of classical chaos.

Here, we present simple models of $1/f$ noise which may essentially influence the understanding of the origin and main properties of the effect. Our models are the result of a search for necessary and sufficient conditions for the appearance of $1/f$ fluctuations in simple systems affected by the random external perturbations, which where initiated in Refs. [1,2] and originated from the observation of the transition from chaotic to nonchaotic behavior in the ensemble of randomly driven systems [3,4].

2. Simple model

The simplest version of our model corresponds to one particle moving along some orbit. The period of this motion fluctuates (due to external random perturbations of the system's parameters) about some average value $\bar{\tau}$. So, a sequence of the transit times $\{t_k\}$ when the particle crosses some point of the orbit is described by the iterative equations

$$\begin{cases} t_k = t_{k-1} + \tau_k, \\ \tau_k = \tau_{k-1} - \gamma(\tau_{k-1} - \bar{\tau}) + \sigma\varepsilon_k. \end{cases} \quad (1)$$

Here, γ is the coefficient of the relaxation of the period to the average value $\bar{\tau}$, $\{\varepsilon_k\}$ denotes the sequence of random variables with zero expectation and unit variance and σ is the standard deviation of the noise. Due to the

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contribution of a large number of random variables to the transit times, model (1) represents a long-memory random process.

Power spectral density $S(f)$ of the signal or current of the model (1), $I(t) = a \sum_k \delta(t - t_k)$ (with a being a contribution to the signal of one pulse or contribution to the current of one particle when it crosses the section of observation), may be calculated according to equation

$$S(f) = \lim_{T \to \infty} \left\langle \frac{2a^2}{T} \left| \sum_{k=k_{\min}}^{k_{\max}} e^{-i2\pi f t_k} \right|^2 \right\rangle, \qquad (2)$$

where T is the whole observation time interval, $k_{\min}$ and $k_{\max}$ are minimal and maximal values of index k in the interval of observation and the brackets $\langle \cdots \rangle$ denote the averaging over the realizations of the process. Eq. (2) may be rewritten in the form

$$S(f) = \lim_{T \to \infty} \left\langle \frac{2a^2}{T} \sum_{k,q} e^{i2\pi f \Delta(k;q)} \right\rangle, \qquad (3)$$

where $\Delta(k;q) \equiv t_{k+q} - t_k$ is the difference of transit times t_{k+q} and t_k.

At $k \gg \gamma^{-1}$, we have the stationary process [5,6]: the expectation $\langle \tau_k \rangle = \bar{\tau}$ and the variance $\sigma_\tau^2 \equiv \langle \tau_k^2 \rangle - \langle \tau_k \rangle^2 = \sigma^2/2\gamma$ of the recurrence time τ_k do not depend on the time while $\Delta(k;q)$ is a normally distributed random variable with the expectation $\mu_\Delta(q) \equiv \langle \Delta(\infty;q) \rangle = q\bar{\tau}$ and the variance $\sigma_\Delta^2(q) \equiv \langle \Delta(\infty;q)^2 \rangle - \langle \Delta(\infty;q) \rangle^2$ expressed as [5,6]

$$\sigma_\Delta^2(q) = \left(\frac{\sigma}{\gamma} \right)^2 \left[q - \frac{(1 - (1 - \gamma)^q)}{\gamma} \right]. \qquad (4)$$

For the normal distribution of $\Delta(k;q) = \Delta(q)$, Eq. (3) yields

$$S(f) = 2\bar{I}a \sum_q e^{i2\pi f \mu_\Delta(q) - 2\pi^2 f^2 \sigma_\Delta^2(q)}, \qquad (5)$$

where $\bar{I} = \lim_{T \to \infty} a(k_{\max} - k_{\min} + 1)/T = a/\bar{\tau}$ is the average current.

Substitution of the expansions of the variance (4) at $|q| \ll \gamma^{-1}$ in powers of $\gamma|q|$,

$$\sigma_\Delta^2(q) = \sigma_\tau^2 q^2,$$

into Eq. (5) yields $1/f$-like power spectrum,

$$S(f) = \bar{I}^2 \frac{\alpha_H}{f}, \qquad (6a)$$

for sufficiently small parameters σ and γ in any desirably wide range of frequencies, $f_1 = \gamma/\pi\sigma_\tau < f < f_2 = 1/\pi\sigma_\tau$. Here α_H is a dimensionless constant (the Hooge parameter),

$$\alpha_H = \frac{2}{\sqrt{\pi}} K e^{-K^2}, \qquad K = \frac{\bar{\tau}}{\sqrt{2}\sigma_\tau}. \qquad (7)$$

We see that the power of $1/f$ noise except the squared average current $\bar{I}^2$ depends strongly on the ratio of the average recurrence time $\bar{\tau}$ to the standard deviation of the recurrence time σ_τ.

Therefore, the process (1) containing only one relaxation time γ^{-1} can for sufficiently small damping γ produce an exact $1/f$-like spectrum in a wide range of frequency (f_1, f_2), with $f_2/f_1 \simeq \gamma^{-1}$.

The model is free from the unphysical divergency of the spectrum at $f \to 0$. So, using for $f < f_1$ an expansion of expression (4) at $|q| \gg \gamma^{-1}$, $\sigma_\Delta^2(q) = (\sigma/\gamma)^2(|q| - 1/\gamma)$, we obtain from Eq. (5) the Lorentzian power spectrum [5,6]

$$S(f) = \bar{I}^2 \frac{4\tau_{\text{rel}}}{1 + \tau_{\text{rel}}^2 \omega^2}. \qquad (8)$$

Here $\omega = 2\pi f$ and $\tau_{\text{rel}} = D_t = \sigma^2/2\bar{\tau}\gamma^2$ is the "diffusion" coefficient of the time t_k. For $f \ll f_0 = \bar{\tau}\gamma^2/\pi\sigma^2 = 1/2\pi\tau_{\text{rel}}$ we have the white noise

$$S(f) = \bar{I}^2 \left(2\sigma^2/\bar{\tau}\gamma^2 \right), \quad f \ll \min\{f_1, f_0\}.$$

This is in agreement with the statement [7] that the power spectrum of any pulse sequence is white at low enough frequencies.

Eqs. (6a)–(8) describe quite well the power spectrum of the random process (1) for perturbation by the Gaussian white noise and even for perturbations by the non-Gaussian sequence of random impacts $\{\varepsilon_k\}$ in Eq. (1) (illustrative examples in Refs. [5,8]).

3. Generalizations and numerical analysis

This simple exactly solvable model can easily be generalized in different directions: for large number of particles moving in similar orbits with coherent (identical for all particles) or independent (uncorrelated for different particles) fluctuations of the periods, for non-Gaussian or continuous perturbations of the systems' parameters and for spatially extended systems. So, when an ensemble of N_c particles moves on closed orbits and the period of each particle fluctuates independently, (due to the perturbations by uncorrelated sequences of random variables $\{\varepsilon_k^v\}$, different for each particle v) the power spectral density of the collective current I of all particles can be calculated by the above method [5] too and it is expressed as the Hooge formula [9,10]

$$S(f) = \bar{I}^2 \frac{\alpha_H}{N_c f}, \qquad N_c = V n_c. \qquad (6b)$$

This expression with the factor $1/N_c$ is in agreement with the Hooge [10,11] statements that summation of spectra is only allowed if the processes contributing to the spectrum are isolated from each other and that only isolated traps yield $1/f$ spectrum.

It should be noted that in many cases the intensity of signals or currents can be expressed as a sequence of the pulse occurrence times $\{t_k\}$, i.e., as $I(t) = a \sum_k \delta(t - t_k)$. This expression represents exactly the flow of identical point objects: cars, electrons, photons and so on. More generally, instead of the Dirac delta function one should introduce time dependent pulse amplitudes $A_k(t - t_k)$. The low frequency power spectral density, however, depends weakly on the shapes of the pulses [7], while fluctuations of the pulses amplitudes result, as a rule, in white or Lorentzian, but not $1/f$, noise. Model (1) in such cases represents fluctuations of the averaged interevent time τ_k between the subsequent occurrence times of the pulses.

The model may also be generalized for the nonlinear relaxation of the interevent time τ_k. In such a case Eq. (1) can be written in the form

$$\begin{cases} t_k = t_{k-1} + \tau_k, \\ \tau_k = \tau_{k-1} - \frac{dV(\tau_{k-1})}{d\tau_{k-1}} + \sigma\varepsilon_k. \end{cases} \quad (9)$$

Here, the function $V(\tau_k)$ represents the effective "potential well" for the Brownian motion of the interevent time τ_k. The steady state distribution density of the interevent time τ_k generated by Eq. (9) is of the form

$$\psi_\tau^{st}(\tau_k) = C \exp\left[-\frac{2V(\tau_k)}{\sigma^2} \right], \quad (10)$$

where a constant C may be obtained from the normalization. For the power-law "potential well"

$$V(\tau_k) = \frac{1}{2}\gamma(\tau_k - \bar\tau)^{2n} \quad (11)$$

with integer n we have a generalization of Eq. (1)

$$\begin{aligned} t_k &= t_{k-1} + \tau_k, \\ \tau_k &= \tau_{k-1} - \gamma n(\tau_{k-1} - \bar\tau)^{2n-1} + \sigma\varepsilon_k. \end{aligned} \quad (12)$$

The steady state distribution density of the interevent time τ_k in such a case is

$$\psi_\tau^{st}(\tau_k) = \frac{(\gamma/\sigma^2)^{1/2n} \exp\left[-\frac{\gamma(\tau_k - \bar\tau)^{2n}}{\sigma^2} \right]}{2\Gamma(1 + 1/2n)}. \quad (13)$$

For sufficiently large $n \gg 1$ Eqs. (12) and (13) represent Brownian motion of the interevent time τ_k in almost rectangle "potential well" (square-well potential) restricting movement of τ_k mostly in the interval $(\bar\tau - h, \bar\tau + h)$ with $h \simeq (\sigma^2/\gamma)^{1/2n}$. At $h < \bar\tau$, such a restriction prevents the emergence of the negative interevent times τ_k and, consequently, from the clustering of the particles or of the signal pulses.

We can evaluate the power spectrum of the processes (9) and (12) as well [5,6,12]. The power spectral density (3) may be rewritten in the form

$$S(f) = 2\bar I a \left\langle \sum_q e^{i2\pi f \tau_k(q)q} \right\rangle, \quad (14)$$

where the occurrence times t_{k+q} and t_k difference $\Delta(k; q)$ is expressed as

$$\Delta(k; q) \equiv t_{k+q} - t_k = \sum_{l=k+1}^{k+q} \tau_k = \tau_k(q)q, \quad q \geqslant 0$$

with $\tau_k(q) \equiv (t_{k+q} - t_k)/q$ being the average interevent time in the time interval (t_k, t_{k+q}) and the brackets in Eq. (14) denote averaging over time (index k) and over realizations of the process. At $2\pi f \tau_k(q) \ll 1$ the summation in Eq. (14) may be replaced by the integration

$$S(f) = 2\bar I a \int_{-\infty}^{+\infty} \langle e^{i2\pi f \tau q} \rangle \, dq.$$

Here, the averaging over k and over realizations of the process coincides with the averaging over the distribution of the interevent times τ_k, i.e.,

$$\langle e^{i2\pi f q \tau} \rangle = \int_{-\infty}^{+\infty} e^{i2\pi f q \tau} \psi_\tau(\tau) d\tau = \chi_\tau(2\pi f q),$$

where $\psi_\tau(\tau_k)$ is the distribution density of the interevent times τ_k and $\chi_\tau(\vartheta)$ is the characteristic function of this distribution.

Taking into account the property of the characteristic function

$$\int_{-\infty}^{+\infty} \chi_\tau(\vartheta) d\vartheta = 2\pi\psi_\tau(0),$$

we obtain the final expression for the power spectrum

$$S(f) = 2\bar I^2 \bar\tau \psi_\tau(0)/f. \quad (15)$$

For Gaussian distribution of the interevent times τ_k,

$$\psi_\tau(0) = \exp\left(-\bar\tau^2/2\sigma_\tau^2 \right) \Big/ \sqrt{2\pi}\sigma_\tau,$$

and according to Eq. (15), we recover expressions (6) and (7) obtained from the analysis of the dynamical process.

Therefore, according to Eqs. (6a), (7) and (15), the dimensionless parameter $\alpha_H = 2\bar\tau\psi_\tau(0)$ is proportional to the distribution density $\psi_\tau(0)$ of the interevent time τ_k in the point $\tau_k = 0$, i.e., to the probability of the clustering of the signal pulses. The pulse clustering results in the large variance of the signal – the condition necessary for appearance of stationary $1/f$ noise in the wide range of frequency.

It should be noted, however, that Eq. (15) represents an idealized $1/f$ noise. The real systems have finite relaxation time and, therefore, expression of the noise intensity in the form of Eqs. (14) and (15) is valid only for $f > (2\pi\tau_{rel})^{-1}$ with τ_{rel} being the relaxation time of the interevent time's τ_k fluctuations. On the other hand,

due to the deviations from the approximation $t_{k+q} - t_k = \tau_k q$ at large q, for sufficiently low frequency we can obtain the finite intensity of $1/f^\delta$ ($\delta \simeq 1$) noise even in the case $\psi(0) = 0$ but for the signals with fluctuations resulting in the dense concentrations of the pulse occurrence times t_k.

We can generate, of course, the stationary time series of the occurrence times t_k also for other restrictions for the interevent time τ_k, e.g., with the reflecting boundary conditions at some values $\tau_k = \tau_{min}$ and $\tau_k = \tau_{max}$. The process like Eq. (1) with the reflecting condition for τ_k at $\tau_{min} = \tau_s$ may also be generated by the recurrent equations

$$\begin{cases} t_k = t_{k-1} + \tau_s + \tau_k, \\ \tau_k = |\tau_{k-1} - \gamma(\tau_{k-1} - \bar{\tau}) + \sigma\varepsilon_k|. \end{cases} \tag{16}$$

Numerical analysis of the models like Eqs. (1), (9), (12) and (16) as well as with other restrictions for the interevent time τ_k shows that power spectrum of the current is $1/f$-like in large interval of frequency only when the distribution density of the interevent times τ_k in the point $\tau_k \simeq 0$ is nonzero, i.e., $\psi_\tau(\tau_k \simeq 0) \neq 0$. For models with $\psi_\tau(0) = 0$ or $\psi_\tau(0)$ very close to zero, we observe in numerical simulations the power spectrum $S(f) \propto 1/f^{3/2}$ (Ref. [8] and Fig. 1).

4. Conclusions

Simple analytically solvable models of $1/f$ noise are proposed. The models reveal main features, parameter dependences and possible origin of $1/f$ noise, i.e., random increments of the time intervals between the pulses or interevent times of the elementary events of the process. The conclusion that $1/f^\delta$ noise with $\delta \simeq 1$ may result from the clustering of the signal pulses, particles or elementary events can be drawn from the analysis of the simple, exactly solvable models. The mechanism of the clustering depends on the system.
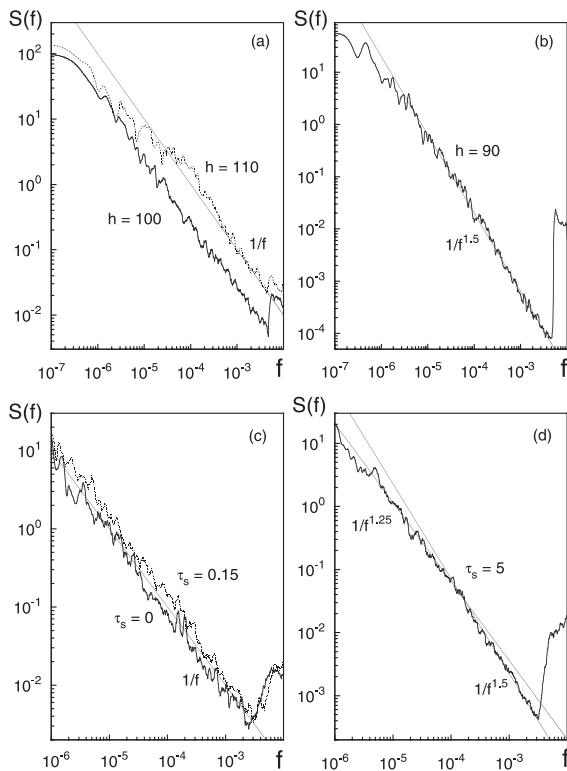
Fig. 1. Power spectrum vs. frequency of the current generated by Eqs. (1)–(3) and (16) with parameters $\bar{\tau} = 100$, $\sigma = 1$ and with the Gaussian distribution of the random increments $\{\varepsilon_k\}$. The sinuous curves represent the results of numerical simulations: (a) and (b) according to Eq. (1) with $\gamma = 0$ and with reflecting boundary conditions for τ_k at $\bar{\tau} - h$ and at $\bar{\tau} + h$ for different values of h; (c) and (d) according to Eq. (16) with $\gamma = 0.0001$ and different values of τ_s.

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