Long-range stochastic point processes with the power-law statistics

Bronislovas Kaulakys, Miglius Alaburda, Vygintas Gontis

Abstract: We analyze the point processes generated by the autoregressive equation for the interevent time. The correlated point process with the Poissonian-like distribution of the time between the neighboring events results in the $1/f$ fluctuations and the power-law distribution of the signal. This is in contrast to the white shot noise and Gaussian distribution of the true Poisson process. The model may be used for the analysis of different long-range processes with the power-law statistics.

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1 Introduction

The power spectra of fluctuations of a large variety of systems ranging from astrophysics and technology to sociology and psychology exhibit "$1/f$ noise", i.e., at low frequencies f the power spectral density of fluctuations behaves as $S(f) \sim 1/f^{\beta}$, where the exponent β is close to 1. Both time-dependent phenomena and spatial series may show such characteristics against the frequency. Great efforts have been made to explain and model the universal presence of $1/f$ noise (see, e.g., comprehensive bibliography of $1/f$ noise in the website [1], review articles [2, 3] and references in the recent papers [4, 5]).

Usually $1/f$ noise theories are formulated for the intensity of the currents or signals. In such cases one starts from the systems of sufficiently complicated, as a rule nonlinear, differential equations with partial derivatives or from the system of equations with a wide and specific distribution of times of the linear relaxations of the signal components. In such a way the obtained signals are, as a rule, Gaussian. However, not all signals exhibiting $1/f$ noise are Gaussian. Some of them are non-Gaussian, exhibiting power-law or even fractal distributions.

In contrast to the Brownian motion generated by the linear stochastic equation, the simple systems of differential, even linear stochastic equations generating signals with $1/f$ noise are not known. Therefore, usually the mathematical models and algorithms for the generation of processes with $1/f$ noise also expose some shortcomings: they are very specific, formal (like "fractional Brownian motion") or unphysical. They cannot usually be solved analytically, and they do not reveal

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either the origin or the necessary and sufficient conditions for the appearance of $1/f$-type fluctuations. This makes the problem of the omnipresent $1/f$ noise one of the oldest problems and puzzles in the contemporary physics.

Some random phenomena, however, occur at discrete times or locations with the individual events largely identical and can be represented as the point processes. A stochastic point process is a mathematical construction which represents these events as random points in space or time. Point processes arise in different fields, such as physics, economics, cosmology, ecology, neurology, the Internet, signaling and telecoms networks and seismology, i.e., in a large variety of systems with the flow of point objects (electrons, photons, cars, pulses, events, and so on) or subsequent actions, like seismic events, neural action potentials, transactions in the financial markets, human heart beats, biological ion-channel openings, burst errors in many communication systems, the Internet network packets, etc.

The complete characterization of a stochastic process involves a description of all possible joint probabilities of various events occurring in the process. Different statistical characteristics provide complementary views of the process. One single statistical characteristic cannot in general describe a stochastic process completely. Fractal stochastic processes exhibit scaling in their statistics. Fractal stochastic point processes exhibit scaling in all statistics, while the fractal-rate stochastic point processes are endowed with rate functions that are either fractal themselves or their increments are fractal [6].

$1/f$ noise, or $1/f$ fluctuations are usually related with the power-law distributions of other statistics of the fluctuating signals, first of all with the power-law decay of autocorrelations, with the power-law distribution of the signal intensities and with the long-memory processes.

It is the purpose of this paper to present analytical and numerical results of the modeling of flows represented as a correlated non-Poissonian, however with the Poissonian-like distribution of the time between the neighboring events, point process resulting in $1/f$ noise and power-law distribution of the signal intensity. Such Poissonian-like processes may exhibit the power-law distribution of the signal intensity and the truncated power-law distributions of the counting statistics, i.e., the phenomena observable in different systems, including the financial one (see, e.g., paper [7] and references herein, as well as references in papers [5, 8]).

2 Point process

In many cases the intensity of the fluctuating signals or currents can be represented by a sequence of random (however, as a rule, mutually correlated) pulses or elementary events $A_k(t - t_k)$,

$$I(t) = \sum_k A_k(t - t_k) \tag{1}$$

where the function $A_k(\phi)$ represents the shape of the k pulse making an influence on the signal $I(t)$ in the region of the pulse occurrence time t_k.

Fluctuations of the shape and of the magnitude of the pulses $A_k(t - t_k)$ usually are uncorrelated or without the long-range correlations. Consequently, they do not result in the low frequency noise and the power-law decay of the autocorrelations. Therefore, here we will restrict our analysis to the fluctuations due to the correlations between the occurrence times t_k. In such approach we can replace the function $A_k(t - t_k)$ by the Dirac delta function and then express the signal as

$$I(t) = \bar{a} \sum_k \delta(t - t_k) \tag{2}$$

with $\bar{a}$ being an average contribution to the signal of one pulse. This model also corresponds to the flow of identical objects: electrons, photons, cars, and so on, and is called the point process model.

The point process is completely described by the set of event times $\{t_k\}$ or, equivalently, by the set of interevent, interpulse intervals $\tau_k = t_{k+1} - t_k$. Such point processes might be called fractal if some relevant statistical characteristics display scaling, characterized by a power-law behavior, with related scaling coefficients indicating that the phenomena contain clusters of points over a relatively large set of time scales.

The power spectrum of the point process signal is described completely by the set of the correlated interevent intervals $\tau_k = t_{k+1} - t_k$. Moreover, the low frequency noise is defined by the statistical properties of the signal at a large-time-scale, i.e., by the fluctuations of the time difference

$$\Delta(k;q) \equiv t_{k+q} - t_k = \sum_{i=k}^{k+q-1} \tau_i \tag{3}$$

at large q, determined by the slow dynamics of the *mean* (average) interpulse time between the occurrence of pulses k and $k + q$ with $q \gg 1$.

Quite generally the dependence of the mean interevent time $\tilde{\tau}_k$ may be described by the general Langevin equation. The Langevin equation may be written down in the actual time t or, equivalently, in the space of the occurrence numbers k with the drift coefficient $h(\tilde{\tau}_k)$ and a multiplicative noise $g(\tilde{\tau}_k)\xi(k)$,

$$\frac{d\tilde{\tau}_k}{dk} = h(\tilde{\tau}_k) + g(\tilde{\tau}_k)\xi(k). \tag{4}$$

Here we interpret k as a continuous variable while the white Gaussian noise $\xi(k)$ satisfies the standard condition

$$\langle \xi(k)\xi(k') \rangle = \delta(k - k') \tag{5}$$

with the brackets $\langle \ldots \rangle$ denoting the averaging over the realizations of the process.

We will use the Itô definition [9] of the stochastic equations.

Transition from the occurrence numbers k to the actual time t in Eq. (4) may be fulfilled using the relation $dt = \tilde{\tau}_k dk$ [10, 11].

The actual sequence of the interevent times τ_k may be superimposed by some additional noise or stochasticity, e.g., τ_k may be determined by the "Poisson" distribution

$$P(\tau_k) = \frac{1}{\tilde{\tau}_k} e^{-\tau_k/\tilde{\tau}_k} \tag{6}$$

with the slowly, according to Eq. (4), or similarly variable the mean interevent time $\tilde{\tau}_k$. Such additional stochasticity does not influence the long-range statistical properties and the low frequency spectra of the process. Therefore, further we will restrict the analysis to the processes generated by Eq. (4) and will identify τ_k with $\tilde{\tau}_k$.

3 Power spectral density

The power spectral density of the point process may be entirely defined by the occurrence times t_k and expressed as

$$S(f) = \lim_{T\to\infty} \left\langle \frac{2}{T} \left| \int_{t_i}^{t_f} I(t) e^{-i\omega t} dt \right|^2 \right\rangle = \lim_{T\to\infty} \left\langle \frac{2\bar{a}^2}{T} \sum_k \sum_{q=k_{\min}-k}^{k_{\max}-k} e^{i\omega\Delta(k;q)} \right\rangle \tag{7}$$

where t_i and t_f are initial and final observation times, $T = t_f - t_i \gg \omega^{-1}$ is the whole observation time and $\omega = 2\pi f$ is the cyclic frequency. Here $k_{\min}$ and $k_{\max}$ are minimal and maximal values of index k in the interval of observation T, the quantity $\Delta(k;q)$ is defined by Eq. (3) and the brackets $\langle \ldots \rangle$ denote the averaging over the realizations of the process.

We have proposed [12, 13, 14] the autoregressive in time axis model for the interevent times. The generalization of it is the multiplicative autoregressive model [5, 8] described by the recurrent equation

$$\tau_{k+1} = \tau_k + \gamma\tau_k^{2\mu-1} + \sigma\tau_k^{\mu}\varepsilon_k \tag{8}$$

for the interevent time. Here γ represents the nonlinear relaxation of the signal $I \simeq \bar{a}/\tau$, while τ_k fluctuates due to the perturbation by normally distributed uncorrelated random variables ε_k with a zero expectation and unit variance and σ is a standard deviation of the white noise. Eq. (8) is the difference (discrete) version of the differential equation (4) with the nonlinear drift $h(\tau_k) = \gamma\tau_k^{2\mu-1}$ and the multiplicative noise $\sigma\tau_k^{\mu}\varepsilon_k$, resulting in $1/f^{\beta}$ noise and the power-law steady-state [9] distribution, $P_k(\tau_k) \sim \tau_k^{\alpha}$, of the interevent time τ_k with the exponent $\alpha = 2\gamma/\sigma^2 - 2\mu$ [5, 8].

Performing the numerical simulations according to Eq. (8) one should restrict in some way the diffusion of the interevent time in some interval $(\tau_{\min}, \tau_{\max})$ [5, 8].

The power spectrum for the process (8), when $\gamma/(\pi\tau_{\max}^{2-\delta})\boldsymbol{lfl}\gamma/(\pi\tau_{\min}^{2-\delta})$, is [5, 8]

$$S(f) \sim \frac{1}{f^{\beta}} \tag{9}$$

where

$$\beta = 1 + \frac{\alpha}{3-2\mu}, \quad \frac{1}{2} < \beta < 2. \tag{10}$$

For $\mu = 1$ we have a completely multiplicative point process when the stochastic change of the interpulse time is proportional to itself. Another case of interest concerns $\mu = 1/2$, when we have the Brownian motion of the interevent time in the actual time with the linear relaxation of the signal $I \simeq \bar{a}/\tau$ and the additive noise,

$$\frac{d\tau}{dt} = \gamma\frac{1}{\tau} + \sigma\xi\left(t\right), \tag{11}$$

(see Refs [5, 8] for details).

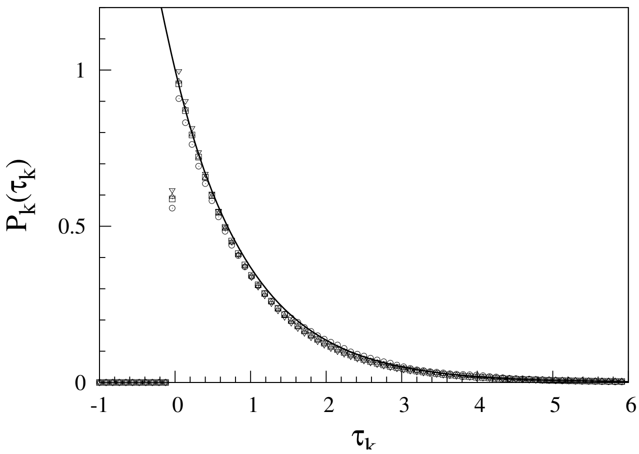

Figure 1: Distribution density of the interevent time τ_k generated according to Eq. (13) with $\sigma = 0.01$ and of the added up together $n = 10$, 100 and 1000 independent such processes with the same intensity of the total signal, different symbols. The solid line represents the analytical distribution (12).

4 Poissonian-like process

Here we will consider the model generating the Poissonian-like process, i.e., with the exponential distribution of the interevent time τ_k,

$$P_k\left(\tau_k\right) = e^{-\tau_k} \tag{12}$$

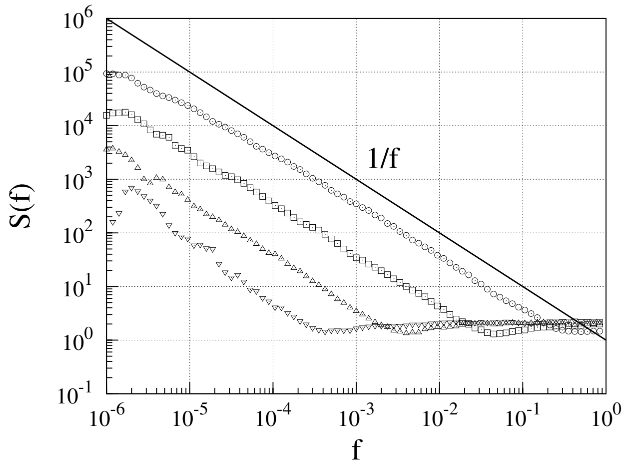

Figure 2: Power spectral density of the Poissonian-like process (2) with $\bar{a} = 1$ generated by Eq. (13), the upper curve represented by the open circles, and of the added up together 10, 100 and 1000 independent such processes with the same intensity of the total signal, the curves represented by different symbols with the decreasing intensity of the noise, respectively.

with the mean interevent time $\tilde{\tau}_k = 1$. Such steady-state distribution may be generated by the particular autoregressive equation (8), i.e.,

$$\tau_{k+1} = \tau_k - \sigma^2/2 + \sigma\varepsilon_k \tag{13}$$

with the reflective boundary condition at $\tau_k = 0$.

The distribution density of the interevent time τ_k generated by this equation is exponential, as of the true Poisson process (see Fig. 1). However, in contrast to the white shot noise of the real Poisson process, the power spectrum of this autoregressive process is $1/f$ (see Fig. 2). In figure 2 the power spectral densities of the added up together $n = 10$, 100 and 1000 independent such processes with the same total average intensity of the flow, $\bar{I}$, i.e., with the resulting average interevent time $\bar{\tau}_k = 1$, generated by Eq. (13) are shown, as well. We see the decrease of the intensity of $1/f$ noise,

$$S(f) \sim \bar{I}^2 \frac{1}{nf}, \tag{14}$$

with increasing of the number of the independent noise sources n, in accordance with the empirical Hooge [15] formula and with the theoretical results of Ref. [14]. The increase of the number n of independent processes moves the resulting process closer the entirely random, uncorrelated, i.e., true Poisson process.

The distribution density of the signal, defined as $I = 1/\tau_k$, is shown in Fig. 3 for the process generated by Eq. (13) and for the the added up together $n = 10$, 100

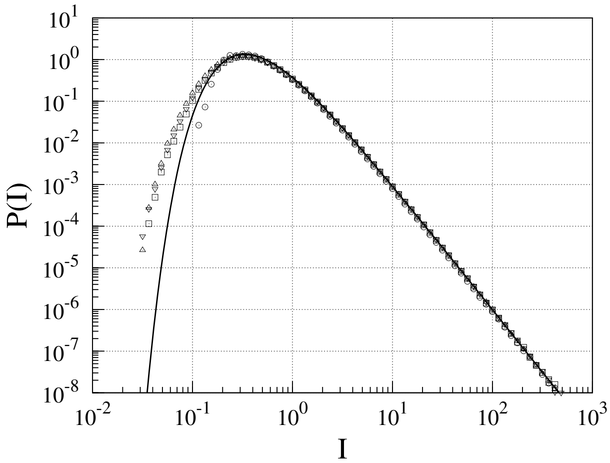

Figure 3: Distribution densities of the signals of the processes (2) with $\bar{a} = 1$ described in captures to Fig. 1 and Fig. 2. The solid curve corresponds to Eq. (15).

and 1000 independent processes generated by Eq. (13), as well. We see that the distribution density of all processes is the same and may be described be equation

$$P(I) = \exp\left\{-\frac{1}{I}\right\}\frac{1}{I^3}, \tag{15}$$

following from Eq. (13) after transformation of the variable (see paper [5] for details). This is essentially the power-law, $P(I) \sim 1/I^3$, distribution for large $I \gg \bar{I}$.

The distribution density of the counts, i.e., of the number N of events [6] in some time interval Δt for the pure Poisson process is

$$P_p(N) = \frac{\bar{N}^N}{N!}e^{-\bar{N}} \tag{16}$$

with $\bar{N} = \Delta t/\bar{\tau}_k$ being the average number of events in the time interval Δt. For $\bar{N} \gg 1$ this distribution approaches the Gaussian distribution with the variance equal to the average, $\sigma_N^2 = \bar{N}$.

For the autoregressive process (13) the distribution density $P(N)$ of the counting is essentially different from the Poissonian one $P_p(N)$ (see Fig. 4 where both distribution densities for $\bar{N} = 10$ are shown).

In figure 5 the cumulative distributions

$$P_>(N) = \sum_{i=N}^{\infty} P(i) \tag{17}$$

of the counting of the point process generated by the autoregressive model (13) for the different average number of events $\bar{N} = 1$, 10, 100, 1000 and 10000 are shown together with those of the pure Poisson process.

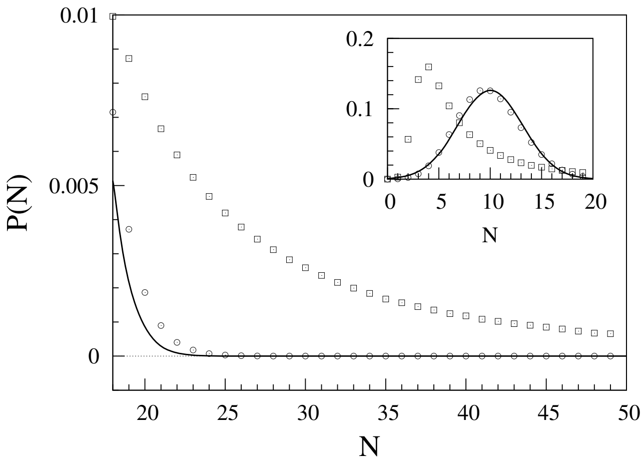

Figure 4: Distribution densities of the counting of events for the pure Poisson process, open circles, and for the process generated by Eq. (13) with $\bar{N} = 10$, open squares. The solid curve corresponds to the Gaussian distribution with $\bar{N} = \sigma_N^2 = 10$.

We see the power-law "fat tails" of the counting events of the autoregressive process with the Poissonian-like distribution (12) of times between the neighboring events.

For the pure Poisson process the cumulative distributions decrease very fast for $N > \bar{N}$, as the error function,

$$P_p(N > \bar{N}) \simeq \frac{1}{2}\mathrm{erfc}(x), \quad x = \frac{N - \bar{N}}{\sqrt{2\bar{N}}}, \tag{18}$$

for $\bar{N} \gg 1$.

We see the huge difference between the probabilities to observe the large deviation of the counting number N from the average $\bar{N}$ of the autoregressive point process comparing to the true Poisson process.

5 Possible generalizations

The possible generalization of the proposed model may be the analysis of the more general equations (4) and (8). We can consider, as well, the more realistic model with the autoregressive change of the mean interevent time $\tilde{\tau}_k$ superimposed by the Poisson distribution (6) of the actual interevent time τ_k. The distribution of the actual interevent time in such a case would be the average of the Poisson distribution (6) over the distribution of the mean interevent time $\tilde{\tau}$ in the time axis,

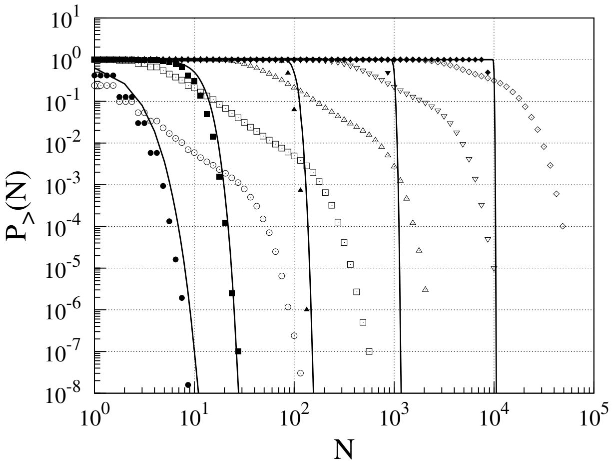

Figure 5: Cumulative distribution of the counting of events for the pure Poisson process, full symbols of the numerical simulations and solid lines of the analytical expressions (16)–(18); and for the autoregressive process (13), open symbols. Results are presented for $\bar{N} = 1$, 10, 100, 1000 and 10000, the increasing probabilities, respectively.

$P_t(\tilde{\tau})$, i.e.,

$$P_t(\tau_k) = \int P_t(\tilde{\tau})\, e^{-\tau_k/\tilde{\tau}} \tilde{\tau}^{-1} d\tilde{\tau}. \tag{19}$$

This additional stochasticity of the interevent time by the randomization of the concrete occurrence times does not influence the low frequency power spectra of the signal.

On the other hand, the stochastic nonlinear differential equations may be derived [10, 11] for the intensity of the signal, $I \simeq \bar{a}/\tau$, starting from the point process models (4) and (8). These equations generate the long-range dependent processes with power-law distributions and $1/f^{\beta}$ fluctuations.

6 Conclusions

We have analysed the stochastic point process with the power-law distribution of the intensity of the signal and $1/f$ noise of the power spectral density generated by the autoregressive equation. The analysis shows that even the Poissonian-like process with the exponential distribution of the interevent time may exhibit scaling and even fractal dependences.

The power-law distributions are observable in different systems from physics, astronomy and seismology to the Internet, financial markets, neural spikes, and human cognition. The analysed model relates the power-law spectral density with the power-law distribution of the signal intensity and may be used for the explanation of different long-range processes (see, e.g., papers [5, 8, 16] and references herein).

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