Tsallis distributions and $1/f$ noise from nonlinear stochastic differential equations

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Probability distributions that emerge from the formalism of nonextensive statistical mechanics have been applied to a variety of problems. In this article we unite modeling of such distributions with the model of widespread $1/f$ noise. We propose a class of nonlinear stochastic differential equations giving both the q-exponential or q-Gaussian distributions of signal intensity, revealing long-range correlations and $1/f^\beta$ behavior of the power spectral density. The superstatistical framework to get $1/f^\beta$ noise with q-exponential and q-Gaussian distributions of the signal intensity is proposed, as well.

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I. INTRODUCTION

Stationary stochastic processes and signals are prevalent across many fields of science and engineering. Many complex systems show large fluctuations of macroscopic quantities that follow non-Gaussian, heavy-tailed, power-law distributions with the power-law temporal correlations, scaling, and the (multi-)fractal features [1–3]. The power-law distributions, scaling, self-similarity, and fractality are sometimes related both with the nonextensive statistical mechanics [4–8] and with the power-law behavior of the power spectral density, i.e., $1/f^\beta$ noise (see, e.g., Refs. [3,9–11], and references herein).

There exist a number of systems, involving long-range interactions, long-range memory, and anomalous diffusion, that possess anomalous properties in view of traditional Boltzmann-Gibbs statistical mechanics. Nonextensive statistical mechanics represents a consistent theoretical background for the investigation of some properties, like fractality, multifractality, self-similarity, long-range dependencies, and so on, of such complex systems [6–8]. Concepts related with nonextensive statistical mechanics have found applications in a variety of disciplines, including physics, chemistry, biology, mathematics, economics, informatics, and the interdisciplinary field of complex systems (see, e.g., Refs. [12–14] and references herein).

The nonextensive statistical mechanics framework is based on the entropic form [4,6]

$$S_q = \frac{1 - \int_{-\infty}^{+\infty} [p(z)]^q dz}{q - 1}, \tag{1}$$

where $p(z)$ is the probability density of finding the system with the parameter z. Entropy (1) is an extension of the Boltzmann-Gibbs entropy $S_{BG} = -\int_{-\infty}^{+\infty} p(z) \ln p(z) dz$, which restores from Eq. (1) at $q = 1$ [6,7].

By applying the standard variational principle on entropy (1) with the constraints $\int_{-\infty}^{+\infty} p(z) dz = 1$ and

$$\frac{\int_{-\infty}^{+\infty} z^2 [p(z)]^q dz}{\int_{-\infty}^{+\infty} [p(z)]^q dz} = \sigma_q^2, \tag{2}$$

where σ_q^2 is the generalized second-order moment [15–17], one obtains the q-Gaussian distribution density

$$p(z) = A \exp_q(-Bz^2). \tag{3}$$

Here $\exp_q(\cdot)$ is the q-exponential function defined as

$$\exp_q(x) \equiv [1 + (1 - q)x]_+^{\frac{1}{1-q}}, \tag{4}$$

with $[(\dots)]_+ = (\dots)$ if $(\dots) > 0$, and zero otherwise. Asymptotically, as $x \to \infty$, $\exp_q(x) \sim x^{-\lambda}$, where $\lambda = (q - 1)^{-1}$, i.e., we have the power-law distribution. The (more) generalized entropies and distribution functions are introduced in Refs. [18,19].

Statistics associated to Eqs. (1)–(4) has been successfully applied to phenomena with the scale-invariant geometry, like in low-dimensional dissipative and conservative maps [20,21], anomalous diffusion [22], turbulent flows [23], Langevin dynamics with fluctuating temperatures [24–26], long-range many-body classical Hamiltonians [27], and financial systems [28,29].

For the modeling of distributions of the nonextensive statistical mechanics, nonlinear Fokker-Planck equations and corresponding nonlinear stochastic differential equations (SDEs) [22,30] and SDEs with additive and multiplicative noises [31,32], with multiplicative noise only [33], and with fluctuating friction forces [26] have been proposed. However, the exhibition of the long-range correlations and $1/f^\beta$ noise has not been observed.

The phrases "$1/f$ noise," "$1/f$ fluctuations," and "flicker noise" refer to the phenomenon, having the power spectral density at low frequencies f of signals of the form $S(f) \sim 1/f^\beta$, with β being a system-dependent parameter. Signals with $0.5 < \beta < 1.5$ are found widely in nature, occurring in physics, electronics, astrophysics, geophysics, economics, biology, psychology, language, and even music [34–38] (see also references in Ref. [10]). The case of $\beta = 1$, or "pink noise", is the one of the most interesting. The widespread occurrence of processes exhibiting $1/f$ noise suggests that a generic, or at least mathematical, explanation of such phenomena might exist.

One common way for describing stochastic evolution and properties of complex systems is by means of generalized stochastic differential equations of motion [39–41]. These nondeterministic equations of motion are used in many systems of interest, such as simulating the Brownian motion in

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statistical mechanics, fundamental aspects of synergetics and biological systems, field theory models, the financial systems, and in other areas [2,39,42,43].

The purpose of this article is to model together both the Tsallis distributions and $1/f$ noise using the same nonlinear stochastic differential equations. The superstatistical approach for modeling of such processes is proposed, as well.

We considered a class of nonlinear stochastic differential equations giving the power-law behavior of the probability density function (PDF) of the signal intensity and of the power spectral density ($1/f^\beta$ noise) in any desirably wide range of frequency. Modification of these equations by introduction of an additional parameter yields Brownian-like motion for small values of the signal and avoids power-law divergence of the signal distribution, while preserving $1/f^\beta$ behavior of the power spectral density. The PDF of the signal generated by modified SDEs is q-exponential or q-Gaussian distribution of the nonextensive statistical mechanics. The superstatistical framework using a fast dynamics with the slowly changing parameter described by nonlinear stochastic differential equations can retain $1/f^\beta$ behavior of the power spectral density as well. When the PDF of the rapidly changing variable is exponential or Gaussian, we obtain q-exponential or q-Gaussian long-term stationary PDF of the signal, respectively.

II. NONLINEAR STOCHASTIC DIFFERENTIAL EQUATION GENERATING ASYMPTOTICALLY POWER-LAW SIGNALS WITH $1/f^\beta$ NOISE

Starting from the point process model, proposed and analyzed in Refs. [9,44–48], the nonlinear SDEs generating processes with $1/f^\beta$ noise are derived [10,49,50]. The general expression for the proposed class of Itô SDEs is

$$dx = \sigma^2 \left(\eta - \tfrac{1}{2}\lambda\right) x^{2\eta-1} dt + \sigma x^\eta dW. \tag{5}$$

Here x is the signal, $\eta \neq 1$ is the power-law exponent of the multiplicative noise, λ defines the behavior of stationary probability distribution, and W is a standard Wiener process (the Brownian motion).

The nonlinear SDE (5) has the simplest form of the multiplicative noise term, $\sigma x^\eta dW$. Equations with multiplicative noise and with the drift coefficient proportional to the Stratonovich drift correction for transformation from the Stratonovich to the Itô stochastic equation [51] generate signals with the power-law distributions [10]. Equation (5) is such an equation. Therefore, the relationship between the exponents in the drift term, $2\eta - 1$, and in the noise term, η, of these equations follows from the requirement of modeling the signals with the power-law distributions. More reasoning of the correlation of these exponents, and of the type of equations like Eq. (5), in general, have been given in Refs. [9,10,49,50,52,53].

On the other hand, the simple transformation of the variable, $y = x^\alpha$, gives an equation of the same type Eq. (5) only with different parameters, $\sigma' = \alpha\sigma$, $\eta' = (\eta - 1)/\alpha + 1$, and $\lambda' = (\lambda - 1)/\alpha + 1$. For example, for $\alpha = 1 - \eta$ with $\eta \neq 1$ we get $\eta' = 0$, i.e., an equation for the variable y having additive noise and nonlinear drift, well known in econophysics and finance SDE to describe the Bessel process [43]. Thus, the observable

x may be a function of another variable y, described by a simpler SDE with additive noise.

Nonlinear SDE, corresponding to a particular case of Eq. (5) with $\eta = 0$, i.e., with linear noise and nonlinear drift, was considered in Ref. [54]. It has been found that if the damping decrease with increasing $|x|$, then the solution of such a nonlinear SDE has long correlation time. The connection of the power spectral density of the signal generated by SDE (5) with the behavior of the eigenvalues of the corresponding Fokker-Planck equation was analyzed in Ref. [55]. This connection was generalized in Ref. [56] where it has been shown that $1/f^\beta$ noise is equivalent to a Markovian eigenstructure power relation.

In order to obtain a stationary process and avoid the divergence of a steady-state PDF the diffusion of stochastic variable x should be restricted at least from the side of small values, or Eq. (5) should be modified. The Fokker-Planck equation corresponding to SDE (5) with restrictions of diffusion of stochastic variable x gives the power-law steady-state PDF

$$P(x) \sim x^{-\lambda} \tag{6}$$

with the exponent λ, when the variable x is far from the ends of the diffusion interval. The simplest choice of the restriction is the reflective boundary conditions at $x = x_{\min}$ and $x = x_{\max}$. Exponentially restricted diffusion with the steady-state PDF

$$P(x) \sim \frac{1}{x^\lambda} \exp\left[-\left(\frac{x_{\min}}{x}\right)^m - \left(\frac{x}{x_{\max}}\right)^m\right] \tag{7}$$

is generated by the SDE

$$dx = \sigma^2 \left[\eta - \frac{1}{2}\lambda + \frac{m}{2}\left(\frac{x_{\min}^m}{x^m} - \frac{x^m}{x_{\max}^m}\right)\right] x^{2\eta-1} dt + \sigma x^\eta dW \tag{8}$$

obtained from Eq. (5) by introducing the additional terms.

In Refs. [9,50] it was shown that SDE (5) generates signals with power spectral density

$$S(f) \sim \frac{1}{f^\beta}, \quad \beta = 1 + \frac{\lambda - 3}{2(\eta - 1)}. \tag{9}$$

in a wide interval of frequencies. SDE (5) exhibits the following scaling property: changing the stochastic variable from x to a scaled variable $x' = ax$ changes the time scale of the equation to $t' = a^{2(1-\eta)}t$ leaving the form of the equation unchanged. This scaling property is one of the reasons for the appearance of the $1/f^\beta$ power spectral density.

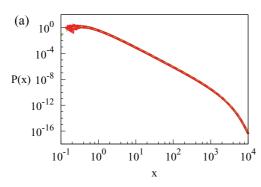For $\lambda = 3$ we get that $\beta = 1$ and SDE (5) should give signal exhibiting $1/f$ noise. One example of the equation (8) with $\lambda = 3$, $m = 1$, $\sigma = 1$, and $\eta = 5/2$ is

$$dx = \left[1 + \frac{1}{2}\left(\frac{x_{\min}}{x} - \frac{x}{x_{\max}}\right)\right] x^4 dt + x^{5/2} dW. \tag{10}$$

Note that $\eta = 5/2$ corresponds to the simplest point process model with the Brownian motion of the interevent time τ_k in the events space (k space) [9,49,50],

$$d\tau_k = \sigma dW_k. \tag{11}$$

Consequently, the simple point process model may provide one possible reasoning of use of the strongly nonlinear
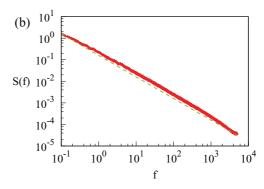
FIG. 1. (Color online) (a) Steady-state PDF $P(x)$ of the signal generated by Eq. (10) with $x_{\min} = 1$ and $x_{\max} = 1000$. The dashed (green) line is the analytical expression for the steady-state PDF. (b) Power spectral density $S(f)$ of the same signal. The dashed (green) line shows the slope $1/f$.

multiplicative SDEs for modeling of long-range correlated systems.

Comparison of numerically obtained steady-state PDF and the power spectral density with analytical expressions is presented in Fig. 1. For the numerical solution we use the Euler-Marujama approximation with the variable step of integration, transforming the differential equations to difference equations [49,50]. We see good agreement of the numerical results with the analytical expressions. A numerical solution of the equations confirms the presence of the frequency region for which the power spectral density has $1/f^\beta$ dependence. The width of this region can be increased by increasing the ratio between the minimum and the maximum values of the stochastic variable x. In addition, the region in the power spectral density with the power-law behavior depends on the exponent η: if $\eta = 1$, then this width is zero; the width increases by increasing the difference $|\eta - 1|$ [55].

The numerical analysis of the proposed SDE (5) reveals the secondary structure of the signal composed of peaks or bursts, corresponding to the large deviations of the variable x from the proper average fluctuations [10]. Bursts are characterized by power-law distributions of burst size, burst duration, and interburst time.

III. STOCHASTIC DIFFERENTIAL EQUATIONS GIVING q-DISTRIBUTIONS

The power spectral density of the form $1/f^\beta$ is determined mainly by power-law behavior of the coefficients of SDEs (5) and (8) at large values of $x \gg x_{\min}$. Changing the coefficients at small x, the spectrum preserves the power-law behavior. In addition, the Fokker-Planck equation corresponding to SDE (5) gives the steady-state PDF with power-law dependence on x as does the q-exponential function for large x. Therefore, SDE (5) can be modified to yield generalized canonical distributions of nonextensive statistical mechanics.

A. q-exponential distribution

The modified stochastic differential equation

$$dx = \sigma^2\left(\eta - \tfrac{1}{2}\lambda\right)(x + x_0)^{2\eta-1}dt + \sigma(x + x_0)^\eta dW \quad (12)$$

with the reflective boundary condition at $x = 0$ was considered in Ref. [10]. The Fokker-Planck equation corresponding to SDE (12) for $x \geqslant 0$ gives the q-exponential steady-state PDF

$$P(x) = \frac{\lambda - 1}{x_0}\left(\frac{x_0}{x + x_0}\right)^\lambda = \frac{\lambda - 1}{x_0}\exp_q(-\lambda x/x_0),$$
$$q = 1 + 1/\lambda. \quad (13)$$

The addition of parameter x_0 restricts the divergence of the power-law distribution of x at $x \to 0$. Equation (12) for small $x \ll x_0$ represents the linear additive stochastic process generating the Brownian motion with the steady drift, while for $x \gg x_0$ it reduces to the multiplicative SDE (5). This modification of the SDE retains the frequency region with $1/f^\beta$ behavior of the power spectral density.

Comparison of numerically obtained steady-state PDF and power spectral density with analytical expressions is presented in Fig. 2. We see a good agreement of the numerical results with the analytical expressions. Numerical solution confirms the presence of the frequency region where the power spectral density has $1/f^\beta$ dependence. The lower bound of this frequency region depends on the parameter x_0.

B. q-Gaussian distribution

The stochastic differential equation

$$dx = \sigma^2\left(\eta - \tfrac{1}{2}\lambda\right)\left(x^2 + x_0^2\right)^{\eta-1}x\,dt + \sigma\left(x^2 + x_0^2\right)^{\eta/2}dW, \quad (14)$$

in contrast to all other equations analyzed in this article, allows negative values of x. This equation was introduced in Refs. [57–59]. The simple case of $\eta = 1$ is used in the model of return in Ref. [33]. Note that $\eta = 1$ does not give $1/f^\beta$ power spectral density. The Fokker-Planck equation corresponding to SDE (14) gives a q-Gaussian steady-state PDF

$$P(x) = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}x_0\Gamma\left(\frac{\lambda-1}{2}\right)}\left(\frac{x_0^2}{x_0^2 + x^2}\right)^{\frac{\lambda}{2}}$$
$$= \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi}x_0\Gamma\left(\frac{\lambda-1}{2}\right)}\exp_q\left(-\lambda\frac{x^2}{2x_0^2}\right),$$
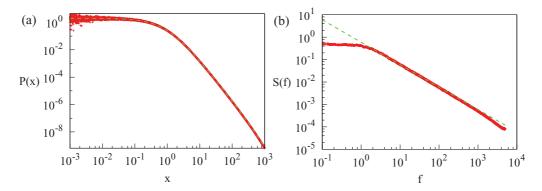$$q = 1 + 2/\lambda. \quad (15)$$

FIG. 2. (Color online) (a) Steady-state PDF $P(x)$ of the signal generated by Eq. (12). The dashed (green) line is the analytical q-exponential expression (13) for the steady-state PDF. (b) Power spectral density $S(f)$ of the same signal. The dashed (green) line shows the slope $1/f$. The parameters used are $\lambda = 3$, $\eta = 5/2$, $x_0 = 1$, and $\sigma = 1$.

The addition of parameter x_0 restricts the divergence of the power-law distribution of x at $x \to 0$. Equation (14) for small $|x| \ll x_0$ represents the linear additive stochastic process generating the Brownian motion with the linear relaxation, while for $x \gg x_0$ it reduces to the multiplicative SDE (5). This modification of the SDE, even the introduction of negative values of the stochastic variable x, does not destroy the frequency region with $1/f^\beta$ behavior of the power spectral density.

Comparison of the numerically obtained steady-state PDF and the power spectral density with analytical expressions is presented in Fig. 3. Good agreement of the numerical results with the analytical expressions is found. Numerical solution confirms the presence of the frequency region where the power spectral density has $1/f^\beta$ dependence.

IV. SUPERSTATISTICS AND $1/f^\beta$ NOISE

Many nonequilibrium systems exhibit spatial or temporal fluctuations of some parameter. There are two time scales: The scale in which the dynamics is able to reach a stationary state and the scale at which the fluctuating parameter evolves. A particular case is when the time needed for the system to reach stationarity is much smaller than the scale at which the fluctuating parameter changes. In the long-term, the nonequilibrium system is described by the superposition of

different local dynamics at different time intervals, which has been called superstatistics [24,60–63]. The superstatistical framework has successfully been applied to a widespread range of problems, such as interactions between hadrons from cosmic rays [25], fluid turbulence [26,64–66], granular material [67], electronics [68], and economics [69–74].

In this article we will consider the case when the fluctuating parameter $\bar{x}$ evolves according to earlier-introduced SDE (8). The parameter $\bar{x}$ changes slowly and can by taken as a constant through a period of time T. Due to the scaling properties of Eq. (5), mentioned in Sec. II, the characteristic time scale in Eq. (5) decreases as a power of x. In order to avoid short time scales and rapid changes of the parameter $\bar{x}$, the possible values of $\bar{x}$ should be restricted from above. If the maximum value of the parameter $\bar{x}$ is $\bar{x}_{\max}$, then the time T during which the parameter $\bar{x}$ changes slowly decreases with increase of $\bar{x}_{\max}$. Within time scale T the signal x has local stationary PDF $\varphi(x|\bar{x})$. The long-term stationary PDF of the signal x is determined as

$$P(x) = \int_0^\infty \varphi(x|\bar{x}) p(\bar{x}) d\bar{x}. \tag{16}$$

We can expect that at small frequencies $\omega \ll T^{-1}$ the spectrum of the signal x is determined mainly by the driving SDE. Therefore, we can get the distribution $P(x)$ determined by Eq. (16) and $1/f^\beta$ power spectral density in a wide region
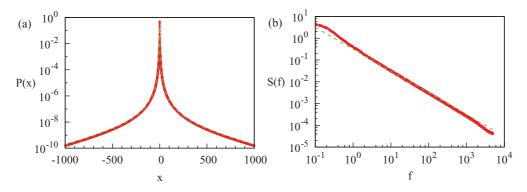

FIG. 3. (Color online) (a) Steady-state PDF $P(x)$ of the signal generated by Eq. (14). The dashed (green) line is the analytical q-Gaussian expression (15) for the steady-state PDF. (b) Power spectral density $S(f)$ of the same signal. The dashed (green) line shows the slope $1/f$. The parameters used are $\lambda = 3$, $\eta = 5/2$, $x_0 = 1$, and $\sigma = 1$.

of frequencies. Using the superstatistical approach, from SDE (8) with the exponential restriction of diffusion, it is possible to obtain the Tsallis probability distributions.

A. *q*-exponential distribution

In order to obtain q-exponential long-term PDF of the signal x we will consider the local stationary PDF conditioned to value of the parameter $\bar{x}$ in the form of exponential distribution

$$\varphi(x|\bar{x}) = \bar{x}^{-1} \exp(-x/\bar{x}). \qquad (17)$$

A Poissonian-like process with slowly diffusing time-dependent average interevent time was considered in Ref. [53]. The mean $\bar{x}$ of the distribution $\varphi(x|\bar{x})$ obeys SDE with exponential restriction of diffusion,

$$d\bar{x} = \sigma^2 \left[\eta - \frac{\lambda}{2} + \frac{1}{2}\frac{x_0}{\bar{x}} - \frac{1}{2}\frac{\bar{x}}{\bar{x}_{\max}} \right] \bar{x}^{2\eta-1} dt + \sigma \bar{x}^\eta dW, \quad (18)$$

where x_0 is a parameter describing exponential cutoff of the steady-state PDF of $\bar{x}$ at small values of $\bar{x}$ and the parameter $\bar{x}_{\max} \gg x_0$ leads to exponential cutoff at large values of $\bar{x}$. When $\bar{x} \ll \bar{x}_{\max}$ the influence of the exponential cutoff at large values of $\bar{x}$ is small. Neglecting $\bar{x}_{\max}$, the steady-state PDF from the Fokker-Planck equation corresponding to Eq. (18) is

$$p(\bar{x}) = \frac{1}{x_0 \Gamma(\lambda - 1)} \left(\frac{x_0}{\bar{x}} \right)^\lambda \exp\left(-\frac{x_0}{\bar{x}} \right). \qquad (19)$$

Using Eqs. (16), (17), and (19), we get that for $x \ll \bar{x}_{\max}$ the long-term stationary PDF of signal x is the q-exponential function

$$P(x) = \frac{\lambda - 1}{x_0} \left(\frac{x_0}{x + x_0} \right)^\lambda = \frac{\lambda - 1}{x_0} \exp_q(-\lambda x/x_0),$$
$$q = 1 + 1/\lambda. \qquad (20)$$

Comparison of numerically obtained long-term PDF and power spectral density with analytical expressions is presented in Fig. 4. A numerical solution confirms the presence of the frequency region where the power spectral density has $1/f^\beta$ dependence. In addition, the long-term PDF of the signal deviates from the q-exponential function (20) only slightly.

B. *q*-Gaussian distribution

In order to obtain the q-Gaussian long-term PDF of the signal x we will consider the local stationary PDF conditioned to the value of the parameter $\bar{x}$ in form of the Gaussian distribution,

$$\varphi(x|\bar{x}) = \frac{1}{\sqrt{\pi}\bar{x}} \exp(-x^2/\bar{x}^2). \qquad (21)$$

The standard deviation of x in the distribution $\varphi(x|\bar{x})$ is proportional to the parameter $\bar{x}$. The fluctuating parameter $\bar{x}$ obeys SDE with exponential restriction of diffusion (8) with the parameter $m = 2$,

$$d\bar{x} = \sigma^2 \left[\eta - \frac{\lambda}{2} + \frac{x_0^2}{\bar{x}^2} - \frac{\bar{x}^2}{\bar{x}_{\max}^2} \right] \bar{x}^{2\eta-1} dt + \sigma \bar{x}^\eta dW, \quad (22)$$

where x_0 is the parameter describing the exponential cutoff of the steady-state PDF of $\bar{x}$ at small values of $\bar{x}$, whereas the parameter $\bar{x}_{\max}(\gg x_0)$ leads to the exponential cutoff at large values of $\bar{x}$. When $\bar{x} \ll \bar{x}_{\max}$, the influence of the exponential cutoff at large values of $\bar{x}$ is small. Neglecting $\bar{x}_{\max}$ the steady-state PDF from the Fokker-Planck equation corresponding to Eq. (22) is

$$p(\bar{x}) = \frac{1}{x_0 \Gamma\left(\frac{\lambda-1}{2}\right)} \left(\frac{x_0}{\bar{x}} \right)^\lambda \exp\left(-\frac{x_0^2}{\bar{x}^2} \right). \qquad (23)$$

From Eqs. (16), (21), and (23) we obtain that for $x \ll \bar{x}_{\max}$ the long-term stationary PDF of the signal x is q-Gaussian, i.e.,

$$P(x) = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi} x_0 \Gamma\left(\frac{\lambda-1}{2}\right)} \left(\frac{x_0^2}{x_0^2 + x^2} \right)^{\frac{\lambda}{2}}$$
$$= \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\sqrt{\pi} x_0 \Gamma\left(\frac{\lambda-1}{2}\right)} \exp_q\left(-\lambda \frac{x^2}{2x_0^2} \right),$$
$$q = 1 + 2/\lambda. \qquad (24)$$

Comparison of the numerically obtained long-term PDF and the power spectral density with analytical expressions is presented in Fig. 5. Numerical solution confirms the presence of the frequency region where the power spectral density has $1/f^\beta$ dependence. In addition, the long-term PDF of the signal deviates only slightly from the q-Gaussian function (24). In contrast to Sec. III B, the superstatistical approach yields $1/f^\beta$ power spectral density only for the absolute value $|x|$ of the
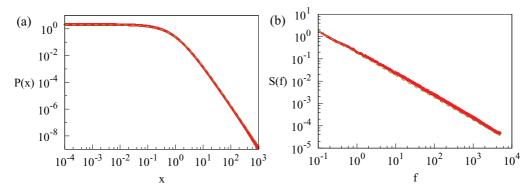

FIG. 4. (Color online) (a) Long-term PDF $P(x)$ of the signal generated by Eqs. (17) and (18). The dashed (green) line is the analytical expression (20) for the long-term PDF. (b) Power spectral density $S(f)$ of the same signal. The dashed (green) line shows the slope $1/f$. The parameters used are $\lambda = 3$, $\eta = 5/2$, $x_0 = 1$, $\sigma = 1$, and $\bar{x}_{\max} = 10^3$.
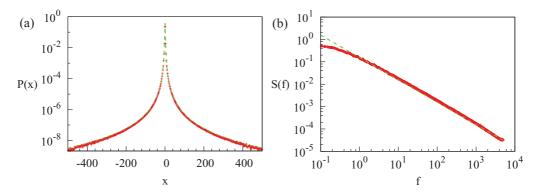
FIG. 5. (Color online) (a) Long-term PDF $P(x)$ of the signal generated by Eqs. (21) and (22). The dashed (green) line is the analytical expression (24) for the long-term PDF. (b) Power spectral density of the same signal. The dashed (green) line shows the slope $1/f$. The parameters used are $\lambda = 3$, $\eta = 5/2$, $x_0 = 1$, $\sigma = 1$, and $\bar{x}_{\max} = 10^3$.

signal. Since the signs of two consecutive values of x are uncorrelated, the spectrum of the signal x itself in the same frequency region is almost flat.

V. DISCUSSION

Common characteristics of complex systems include long-range interactions, long-range correlations, (multi-)fractality, and non-Gaussian distributions with asymptotic power-law behavior. The nonextensive statistical mechanics, which is a generalization of the Boltzmann-Gibbs theory, is a possible theoretical framework for describing these systems. However, long-range correlations and $1/f^\beta$ noise has not been observed in previously used models giving distributions of the nonextensive statistical mechanics. The joint reproduction of the distributions of nonextensive statistical mechanics and $1/f$ noise, presented in this article, extends understanding of the complex systems.

Modeling of the concrete systems by use of nonlinear SDEs is not the goal of this article. However, relation of the models and obtained results with some features of the financial systems can be pointed out. Equations with multiplicative noise are already used to model financial systems, e.g., the well-known 3/2 model of stochastic volatility [75]. There is empirical evidence that trading activity, trading volume, and volatility are stochastic variables with long-range correlations [2,42,43,75–78]. This key aspect, however, is not accounted for in the widely used models. On the other hand, the empirical findings of the PDF of the return and other financial variables are successfully described within the nonextensive statistical framework [12,28,29]. The return has a distribution that is very well fitted by q-Gaussians, only slowly becoming Gaussian as the time scale approaches months, years, and longer times. Another interesting statistic that can be modeled within the nonextensive framework is the distribution of volumes, defined as the number of shares traded. Modeling of some properties of the financial systems using the point process models and SDEs have been undertaken in Refs. [52,58,59,79]. Equations presented in this article incorporating long-range correlations, $1/f^\beta$ noise, and q-Gaussian distributions suggest deeper comprehension of these processes.

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