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RETARDED ACCELERATIONS OF THE SELF-ORGANIZED FRONT: PROPAGATION OF THE BISTABLE FRONT UNDER STEP-LIKE FORCE

A. Raguotis [a], F. Ivanauskas [a,b], and R. Bakanas [c]

[a] *Faculty of Mathematics and Informatics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania*

[b] *Institute of Mathematics and Informatics, Akademijos 4, LT-08663 Vilnius, Lithuania*

[c] *Semiconductor Physics Institute, A. Goštauto 11, LT-01108 Vilnius, Lithuania*

E-mail: bakanas@pfi.lt

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Retarded accelerations of the self-organized front that separates two steady states of a continuous bistable system are studied by considering the response of the "bistable" front (BF) to the step-like force of flexible steepness. The driven system is described by a nonlinear differential equation of reaction–diffusion type, with the rate function approximated by the linear pieces. The retarded response of BF is examined by considering the lag (delay) time between the driving force and the propagation velocity of the driven front. The considered delay time is shown to be sensitive both to the rate (steepness) of the driving force and to the characteristic relaxation time of the system that describes the rate of transient processes within it. At low rates of driving force the response of BF is almost instantaneous. The delay time monotonically increases with increasing the rate (steepness) of the driving force and approaches some fixed value that does not exceed the characteristic relaxation time of the system. The dependence of delay time on the strength of the driving force is weak, insignificant. The derived results evidently show that the ratchet-like transport of BF, previously discussed in [4], should be significantly suppressed in the case of fast driving, when the frequency of applied zero-mean force exceeds the characteristic relaxation rates in the system.

Keywords: nonlinear dynamical systems, pattern formation, diffusion systems, partial differential equations

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1. Introduction

Continuous bistable systems driven far beyond their thermal equilibrium are widely studied as the simplest examples of self-organization. The "bistable" fronts (BFs), the spontaneously formed front structures joining two steady spatially uniform states in a bistable system, are widely known in physically diverse systems and have attracted increasing attention in many branches of physics, chemistry, biophysics, etc. [1]. The free, undisturbed BF always propagates from the state with the higher free energy to the state with the lower free energy. In other words, it propagates in such a way that the more stable state invades the less stable one. The picture changes in the case of the system being acted upon by the temporally oscillating fields, both noisy and regular ones. A topic currently receiving much attention is the unidirectional transport of self-ordered fronts under the temporally oscillating fields of zero mean [2–4]. Two different mechanisms underlying the "unforced" transport of BFs, two different possibilities of the spurious drift have been identified, namely, the parametrically (externally) and directly (internally) induced dc motion of the front. The first, usually described by the "multiplicative" driving force, comes through the action of the external zero-mean field on the externally controllable parameter in the system: the external, symmetrically oscillating field is transformed into the "internal", asymmetrically oscillating forcing, which "pushes" the front in the system (e. g., see [2]). Differently, the directly stimulated dc motion described by the additive force implies that the mean value of the driving force, which acts on the front in the system, equals zero [4]. The response of BF to the additive force $f(t)$ is described by the following equation of reaction–diffusion type:

$$u_t - u_{zz} - cu_z + R(u) = f(t)\,. \qquad (1)$$

Here the function $u(z,t)$ denotes the step-like field of the front propagating at the instantaneous velocity $c(t)$, $z = x - ct$ is the travelling coordinate, and the rate function $R(u)$, which characterizes the rate

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of the transient processes in the system, has three zeroes at $u = u_1, u_2, u_3$ (say, $u_1 < u_2 < u_3$). In the considered case of bistable system one has that $R'(u_{1,3}) > 0$ and $R'(u_2) < 0$, where the prime denotes the derivative. The front solution of the free, undisturbed BF $u_0(z)$ joins two fixed points (steady uniform states) u_1 and u_3, namely, the following relations hold: $u_0(z \to -\infty) \to u_1$, and $u_0(z \to \infty) \to u_3$.

The deterministic version of "front-ratchet", namely, the response of BF to the periodically oscillating force $f(t)$ of zero mean has been recently discussed in [4]. Various types of the unforced dc motion of BFs have been identified within the adiabatic approximation used. The considered approximation implies that the driving force is slow enough, if compared to the characteristic relaxation time of the system τ_{R}, i. e., $T_{\mathrm{F}} \gg \tau_{\mathrm{R}}$, where by T_{F} we denote the characteristic time (period) of the driving force $f(t)$. A rough estimate of the parameter τ_{R}, derived by use of the perturbation technique, was presented in [4]. It reads $\tau_{\mathrm{R}} \cong \min\{R'^{-1}(u_1), R'^{-1}(u_3)\}$, where the prime denotes the derivative.

The response of BF to the slowly varying force is immediate, almost instantaneous on the time scale of the characteristic time T_{F}. In contrast, a delay of response of BF is expected in the case of fast driving, when inequality $T_{\mathrm{F}} < \tau_{\mathrm{R}}$ is fulfilled. In particular, the propagation velocity of the "rapidly" driven front will follow the driving force with some retardation. Clearly, the retardation discussed must influence the "size" of the spurious drift of BF. The decrease of the average drift velocity v of BF is expected with a rapidly oscillating zero-mean force, similarly as in the case of ordinary ratchets [5].

In the present report the delay of response of BF to the fast driving is studied by considering the temporal relaxation of instantaneous velocity of the rapidly driven front being under the action of step-like force of a flexible "profile". We approximate the step-like force by the exponential forcing function $f(t)$ that is characterized by adjustable parameters: the switching time T_{F} and the magnitude F_0 that govern the rate (steepness) and the strength of the driving force, respectively. The main subject of the present study is the speed relaxation time τ_{S} that characterizes the relaxation rate (rapidness) of the instantaneous velocity $c(t)$ of the front being under the action of the step-like force. The "size" of retardation discussed is described by the characteristic delay time τ_{D} defined by the relation $\tau_{\mathrm{D}} := \tau_{\mathrm{S}} - T_{\mathrm{F}}$. The introduced parameter τ_{D} describes the lag time between the driving force $f(t)$ and the instantaneous velocity of the driven front, $c(t)$. In considering the retarded accelerations of BF we derive the needed characteristics $\tau_{\mathrm{S}} - T_{\mathrm{F}}$ and $\tau_{\mathrm{S}} - F_0$ that are studied in a wide interval of parameters T_{F} and F_0, for arbitrary rates and strengths of the driving force. More exactly, the immediate, instantaneous ($\tau_{\mathrm{D}} = 0$) response that takes place in the limit $T_{\mathrm{F}} \to \infty$ is described analytically, within the adiabatic approximation discussed. Differently, the retarded response, the case of fast driving has been examined numerically by direct solution of Eq. (1). By comparing the both discussed cases of the quasi-statically slow and the fast driving that describe instantaneous and retarded response, respectively, we derive the required delay time τ_{D}. The retardation effects in the dynamics of self-organized fronts, as far as we know, have not been considered as yet.

An analytic solution of the governing Eq. (1) with an arbitrary rate function is not feasible even in the case of the free ($f \equiv 0$) system. Thus, when considering the instantaneous response we introduce some simplifying assumptions. The free front solutions of BF have been derived analytically only in a few cases of the rate function approximated by the cubic-polynomial, sine-type and piecewise-linear functions. In the present report the "pseudolinear" model of bistable system is used. We approximate the rate function $R(u)$ by linear pieces, similarly as in the case of "front-ratchet" device (see Refs. [4, 6]). The primary goal of the present study is to present the main outlines of the retarded response that may influence the "size" of the spurious drift. Thus, we deal with the most frequently studied case of the *symmetrical* (symmetrically shaped) rate functions satisfying the relation $R(u_2 - \Delta u) = R(u_2 + \Delta u)$, where the quantity Δu denotes the free variable (see [4]). We note that both the cubic-polynomial and the sine-type rate functions usually used in theoretical studies of the driven fronts are *symmetrical* ones.

In Sect. 2 we discuss the model and the approximations. Section 3 deals with the retarded accelerations of the front being under the action of step-like force. The desired characteristics that describe dependence of the lag time between the driving force and the propagation velocity of BF on both the steepness (rate) and the strength of step-like force are presented. The influence of retardation effect on the spurious drift of BF is briefly discussed, too. Finally, we present the main conclusions.

2. Model, approximations, and techniques used

As noted, the pseudolinear model of bistable system is used. Thus, we write

$$R(u) = \begin{cases} \alpha_1(u-u_1)\,, & u < u_M\,, \\ -\alpha_2(u-u_2)\,, & u_M < u < u_m\,, \\ \alpha_3(u-u_3)\,, & u > u_m\,, \end{cases} \qquad (2)$$

where the free parameters u_i and α_i satisfy the relations $u_M < u_2 < u_m < u_3$ and $\alpha_i > 0$ ($i = 1,2,3$), and the extrema of the rate function, $R_M \equiv R(u_M)$ and $R_m \equiv R(u_m)$, are given by the expression $R_{M,m} = \alpha_2(u_2 - u_{M,m})$. The free ($f \equiv 0$) front solutions of the pseudolinear model have been derived in [6].

Some interesting peculiarities of the spurious drift of BFs have been identified with the periodically oscillating square-pulse force described by a "superposition" of the step-like functions $f(t) = F_0\Theta(t-t_n)$. The square-wave ac force appeared to be very "effective"; the unforced dc drift induced by such an extremely steeply pulsating force is much more strongly pronounced if compared to that in the case of a multiharmonic force (e. g., see [4]). Thus, we approximate the driving force $f(t)$ by the step-like function of a flexible shape:

$$f(t) = \begin{cases} 0\,, & t < t_0\,, \\ F_0\{1-\exp[-\gamma(t-t_0)]\}\,, & t > t_0\,. \end{cases} \qquad (3)$$

Here, as previously, the adjustable parameters γ and F_0 describe the rate (steepness) and the strength (magnitude) of the driving force, respectively, and the quantity t_0 denotes the initial moment at which the driving force was switched on. The obvious relations hold: $T_\mathrm{F} = \gamma^{-1}$ and $f(t;\gamma\to\infty) \to F_0\Theta(t-t_n)$. The limiting case of the quasi-statically slow driving that describes the instantaneous response implies that $\gamma \to 0$ and $\tau_\mathrm{D} \equiv \tau_\mathrm{S} - \gamma^{-1} \to 0$. The required delay time τ_D that characterizes the retarded accelerations of BF may be derived by a direct comparison of relaxation rates of both functions, $c(t;F_0,\gamma\to 0)$ and $c(t;F_0,\gamma)$.

In what follows we shall use the scaled units. Namely, we introduce the scaled "variables" $s(t)$ and $f^*(t)$ defined by the following relations: $s(t) := c(t)/c_\mathrm{P}$ and $f^*(t) := f(t)/\Delta R$, where $c_\mathrm{P} = 2\sqrt{\alpha_2}$ and $\Delta R = R_M - R_m$. It was shown in [4] that the instantaneous velocity $s(t)$ scaled in the units of the marginal velocity c_P of the "pushed" front did not depend on the "height" of the rate function ΔR if the driving force $f(t)$ was taken in the units of the height ΔR. Thus, in what follows we take that $\Delta R = 1$. Hence, it follows that $f^*(t) \equiv f(t)$. Finally, the criterion of the global stability of BF reads: $|F_0| < F_\mathrm{C} \equiv \min\{R_M, -R_m\}$, the strength of the driving force (3) cannot exceed the critical value F_C.

The immediate, instantaneous response takes place in the limit $T_\mathrm{F} \to \infty$. This implies that the lag time between drive and response equals zero; the instantaneous velocity $s(t)$ immediately follows the driving force $f(t)$, without any delay, namely, $\tau_\mathrm{D} = 0$. When considering the case of quasi-statically slow driving we drop the time derivative in Eq. (1). Then we get that

$$u_{zz} + cu_z - R_F[u;f(t)] = 0\,, \quad R_F := R(u) - f(t)\,. \qquad (4)$$

The considered (adiabatic) approximation has already been applied to the front-ratchet "device"; the unforced transport of BF being under the action of a "slowly" oscillating square-wave ac force has been recently discussed in [4]. As already noted, the adiabatic approximation works well for large values of switching time T_F. For the considered case of a pseudolinear rate function the approximate criterion of slow driving reads: $\gamma \ll \tau_R^{-1} \equiv \min\{\alpha_{1,3}\}$.

Quite similarly as in the case of the free system, equation (4), when used in conjunction with the rate function (2), is solvable by the rigorous analytic tools. Both the front solution and the propagation velocity of slowly driven BF may be derived analytically by the direct solution of Eq. (4) used in conjunction with the appropriate boundary and matching conditions. More specifically, the propagation velocity of slowly driven BF is described by the "speed equation" that may be presented in the following manner (see [4]):

$$\frac{Sn(s)}{\exp[-\varphi(s)]\,\sin\Phi(s)} = \frac{h_R - (1+h_R)f(t)}{1+(1+h_R)f(t)}\,, \qquad (5)$$

where the parameter $h_R := -R_M/R_m$ denotes the ratio of extreme values of the rate function, and the auxiliary functions are described by relations

$$\varphi(s) = \frac{s\Phi(s)}{Q_2(s)}\,,$$

$$\Phi(s) = \begin{cases} \arctan Tg(s)\,, & Tg(s) > 0\,, \\ \pi - \arctan[-Tg(s)]\,, & Tg(s) < 0\,, \end{cases}$$

$$Sn(s) = F_{Sn}/F_V\,, \quad Tg(s) = F_{Sn}/F_{Cn}\,. \qquad (6)$$

Here the unknown functions are given by the following expressions:

$$F_{Sn} = Q_2(s)[\delta_1 K_1(s) - \delta_3 K_3(s)]\,,$$

$$F_{Cn} = -\left[Q_2^2(s) + G_1(s)G_3(s)\right],$$

$$F_V = Q_2^2(s) + G_1^2(s)\,,$$

$$G_{1,3} = -s + \delta_{1,3} K_{1,3}(s)\,, \tag{7}$$

where

$$Q_2(s) = \sqrt{1-s^2}\,, \quad K_{1,3} = -s \pm \sqrt{r_{1,3}+s^2}\,. \tag{8}$$

Here and in what follows we use the denotation $r_{1,3} \equiv 1/\delta_{1,3} = \alpha_{1,3}/\alpha_2$. Without loss of generality, the following boundary conditions have been used in the derivation of speed equation (5): $u(z,t) \to v_1(t)$ if $z \to -\infty$, and $u(z,t) \to v_3(t)$ if $z \to \infty$, where the time-dependent quantities $v_{1,3} = u_{1,3} + \alpha_{1,3} f(t)$ denote zeroes of the modified rate function R_{F}. Clearly, equation (4) neglects the retardation. As a consequence, the asymptotic values of the front solution, $u(t; z \to \pm\infty) \to v_{1,3}(t)$, immediately follow the driving force $f(t)$, without any delay. Furthermore, Maxwellian rule holds: the instantaneous velocity of slowly driven BF satisfies the relations $s(t) > 0$ if $S_{\mathrm{F}} > 0$ and $s(t) < 0$ if $S_{\mathrm{F}} < 0$, where the quantity S_{F} denotes the area enclosed by the R_{F}–u characteristic in the interval $[v_1, v_3]$ of the variable u, as shown in [4]. Finally, the velocity $s(t)$ of the quasi-statically driven BF is a function of the relative parameters r_{13} and h_R, but not a function of the absolute values of slope coefficients $\{\alpha_i\}$ and the heights R_M and R_m. Namely, one has that $s(t) = s[r_1, r_3, h_R; f(t)]$. Speed equation (5) describes the instantaneous response and has been used in derivation of the delay time τ_{D} that describes the retarded accelerations of BF.

The retarded response, namely, both the front solution and the instantaneous velocity of the driven BF have been studied numerically, by direct solution of the governing equation (1). The numerical simulations have been used to derive the speed relaxation time τ_{S} for arbitrary rates and strengths of the driving force (3). Let us touch briefly on the most important features of the numerical technique [7]. We introduced a uniform grid to find the numerical solution of Eq. (1). Differential equation (1) was approximated by the finite difference scheme in the co-moving coordinates, $z = x - c(t)\,t$,

$$u_t(z,t) \to h_t^{-1}[u(z_i, t_{j+1}) - u(z_i, t_j)]\,,$$

$$u_{zz}(z,t) \to h_z^{-2}[u(z_{i+1}, t_j) - 2u(z_i, t_j) + u(z_{i-1}, t_j)]\,. \tag{9}$$

Here $h_z > 0$ and $h_t > 0$ are steps of the grid, i and j are integers, and the variables z_i and t_j are described by the relations $z_i = ih_z$ and $t_j = jh_t$. The "size" of the nucleus (kernel) of BF, λ_{F}, which characterizes the spatial extension of the separation wall of BF, is described by the derivative $u_z(z,t)$. Using the co-moving frame we define the centre of "mass" of BF, z_{C}, by the relation

$$z_{\mathrm{C}} = N^{-1} \int_{-L}^{L} \mathrm{d}z\, z\, u_z(z)\,, \quad \text{with}$$

$$N = \int_{-L}^{L} \mathrm{d}z\, u_z(z) \approx v_3 - v_1\,, \tag{10}$$

where the quantity L that indicates the spatial extension of the moving grid satisfies the relation $L \gg \lambda_{\mathrm{F}}$. Hence, the instantaneous velocity of BF is given by the expression

$$c(t_j) \approx c(t_{j-1}) + h_t^{-1}[z_{\mathrm{C}}(t_j) - z_{\mathrm{C}}(t_{j-1})]\,. \tag{11}$$

Taking the limits $L \to \infty$ and $h_t \to 0$ one gets that $c(t_j) \to c(t)$. Further, the speed relaxation time τ_{S} is derived from the following equation:

$$\left|\frac{s_\infty - s(\tau_{\mathrm{S}})}{s_\infty}\right| = \delta s\,, \tag{12}$$

where the quantity $s_\infty := s(t \to \infty) \equiv s(F_0)$ denotes the limiting (extreme) value of the instantaneous velocity of the driven BF, and by δs we denote the relative deviation satisfying the relation $\delta s \ll 1$. The required parameter s_∞ may be derived using both the governing equation (1) and the speed equation (5). Our direct calculations have shown that both discussed values of s_∞, derived from (1) and from (5), coincide within the accuracy of few tenths of percent. Clearly, the speed relaxation time τ_{S} that is defined by relation (8) depends on the parameter δs. To derive the "correct" value δs we have considered the particular case of the slow driving that satisfies the relation $\gamma \ll \alpha_i$. Evidently, the instantaneous velocity of slowly driven BF almost instantaneously follows the driving force $f(t)$, thus, the following relation should be satisfied: $\tau_{\mathrm{S}} \approx T_{\mathrm{F}} \equiv \gamma^{-1}$. The "adequate" value of the deviation δs has been derived by use of both discussed relations. This implies that the discussed parameters τ_{S} and T_{F} practically coincide in the limit of slow driving. Our direct calcula-

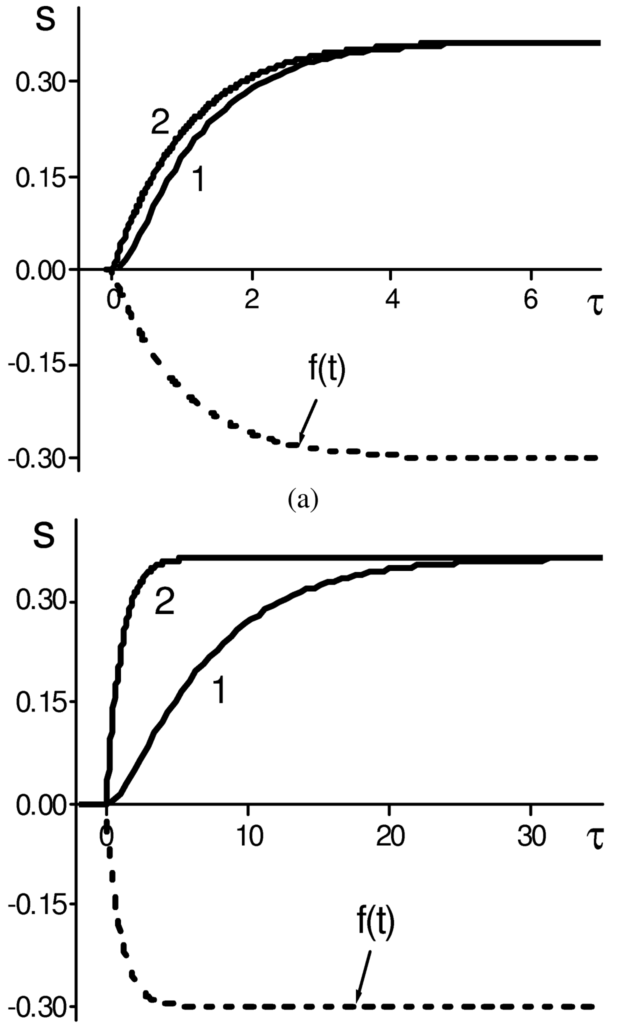

Fig. 1. Relaxation speed of the instantaneous velocity of BF for some fixed values of the parameter α. All curves corresponding to the different values of the parameter, $\alpha = 0.2, 1.0, 5.0$, strictly coincide. The particular cases of (a) slow driving ($\beta = 0.2$), (b) fast driving ($\beta = 5.0$) are presented. Parameter values are $h_R = 1$, $F_0 = -0.3$, and $t_0 = 0$. The results derived from numerical solution of Eq. (1) are shown by curves *1*, and curves *2* show the results received by the use of speed equation (5).

tions carried out by use of Eqs. (1) and (5) have shown that the criterion of slow driving is fulfilled within the accuracy of few percent if one takes that $\gamma \leq 0.1\alpha_i$ (e. g., see Figs. 1 and 2 below).

3. Retarded accelerations of the driven front: Lag time between drive and response

As noted, the particular case of symmetrical rate functions is considered, thus, we take that $\alpha_1 = \alpha_2 = \alpha_3 \equiv \alpha$. For brevity, we introduce the denotation $\beta := \gamma/\alpha$. Now, the criterion of the slow driving reads $\beta \ll 1$, and the limiting case of quasi-static driving implies that $\beta \to 0$.

One may expect that the response of BF will be almost immediate, namely, the lag time between the driving force $f(t)$ and the instantaneous velocity $s(t)$ will be insignificant if the inequality $\beta < 1$ is satisfied. A significant retardation effect is expected in the opposite case of fast driving when the relation $\beta > 1$ holds. The typical s–t characteristics, which describe the cases of both the slow ($\beta = 0.2$) and the fast ($\beta = 5.0$) driving discussed, are shown in Fig. 1. In considering the discussed characteristics we have used the scaled variable, $\tau = \gamma t$; the time is scaled in the units of the switching time T_{F}. Curves *1* in Fig. 1 represent the "exact" result derived from the direct solution of the governing equation (1), whereas s–τ dependences shown by curves *2* have been derived by use of the speed equation (5), within the adiabatic approximation discussed. As expected, the retardation effect is much more pronounced with the fast driving. The response of BF to the slowly varying force ($\beta = 0.2$) is almost instantaneous; both s–τ dependences shown in Fig. 1(a) practically coincide within the accuracy of a few percent. Furthermore, the s–τ characteristics on Fig. 1(a) instantaneously follow the driving force shown by the dashed line in this figure. Quite differently, the considered characteristics presented in Fig. 1(b) ($\beta = 5.0$) exhibit a significant retardation effect. Namely, the s–τ dependence derived by numerical simulations (curve *1*) is much more gently sloped if compared to that obtained by the speed equation (5), within the quasi-static approximation used (curve *2*). Moreover, all the s–τ dependences that have been taken at the different slope coefficients α strictly coincide if the parameter β is taken fixed, as shown by the curves on both Figs. 1(a) and 1(b). The introduced parameter $\beta := \gamma/\alpha$ describes the relative rate (rapidness) of the driving force. Or otherwise, it indicates the ratio of the relaxation rates T_{F}^{-1} and τ_R^{-1} of the driving force and the bistable system, respectively, namely, one has that $\beta = \tau_R/T_{\mathrm{F}}$. Thus, we conclude that the "relative rate" β is the basic parameter that governs the "size" of the delay; the lag time between drive and response increases with the increase of parameter β.

Let us turn to the quantitative characteristics of the accelerations and discuss the required $\tau_{\mathrm{S}}-\gamma$ and $\tau_{\mathrm{S}}-F_0$ dependences that describe the "size" of the retardation. Evidently, the delay time τ_{D} may be derived by considering the speed relaxation time τ_{S}, namely, one has that $\tau_{\mathrm{D}} := \tau_{\mathrm{S}} - T_{\mathrm{F}}$. The typical τ_{S}–γ characteristics derived by the numerical solution of Eq. (1) are shown

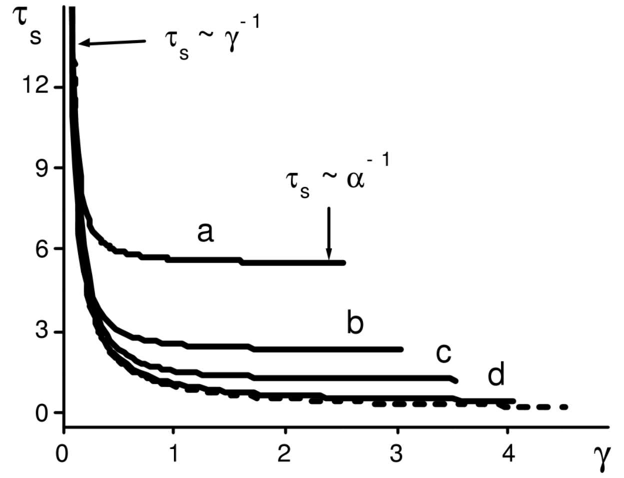

Fig. 2. Dependence of relaxation time τ_{S} on the steepness of the driving force. Solid curves show the results obtained by the use of governing equation (1); dashed curve shows the result received by the use of quasi-static approximation. Parameter values are $h_R = 1$, $F_0 = -0.3$; $\alpha = 0.2$ in curve a; $\alpha = 0.5$ in curve b; $\alpha = 1.0$ in curve c; $\alpha = 3.0$ in curve d.

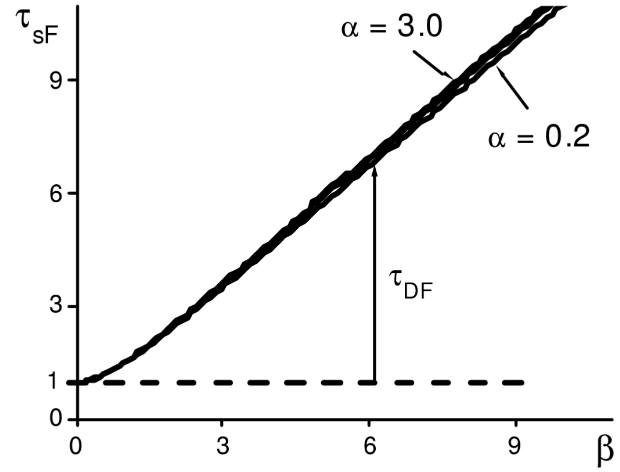

Fig. 3. Dependence of the speed relaxation time τ_{sF} on the parameter β (solid curves); dashed curve shows the result derived by the use of speed equation (5). Parameter values are $h_R = 1$, $\alpha = 0.2, 1.0, 3.0$, and $F_0 = -0.3$.

by curves a, b, c, and d in Fig. 2. All these encompass both previously discussed cases of the slow and the fast driving. The dashed curve derived by speed equation (5) represents the instantaneous response that satisfies the relation $\tau_{\mathrm{S}} = T_{\mathrm{F}} \equiv \gamma^{-1}$. As expected, the relaxation time τ_{S} decreases with the increasing rate (steepness) of the driving force. In the region of small values of γ satisfying the relation $\gamma \ll \alpha$ the considered τ_{S}–γ dependences shown by the curves a, b, c, and d approach the dashed curve that describes the instantaneous response of BF. As a consequence, the delay time τ_{D} tends to zero if the driving force becomes extremely slow. This implies that the adiabatic approximation fits well, and the response of BF may be treated as almost instantaneous one if the inequality $\beta \ll 1$ is satisfied. Differently, in the limiting region of large γ's the considered τ_{S}–γ dependences approach the fixed value $\tau_{\mathrm{S}} \approx \alpha^{-1}$, as shown by the curves a, b, c, and d in Fig. 2. Hence it follows that the delay time satisfies the approximate relation $\tau_{\mathrm{D}} \approx \alpha^{-1} - \gamma^{-1}$ if $\beta > 1$. Furthermore, the presented dependences evidently show that the considered relaxation time τ_{S} taken at some fixed value of the parameter γ increases with the decreasing slope coefficient α. Hence, the delay time τ_{D} decreases with the increasing α.

The considered τ_{S}–γ dependences discussed above have been verified more accurately. With this aim we have introduced the scaled parameters defined by the following relations: $\tau_{\mathrm{sF}} := \tau_{\mathrm{S}}/T_{\mathrm{F}}$ and $\tau_{\mathrm{DF}} := \tau_{\mathrm{D}}/T_{\mathrm{F}}$. The obvious relation that encompasses both discussed cases of the slow and the fast driving holds,

$$\tau_{\mathrm{sF}} \equiv \tau_{\mathrm{DF}} + 1 \approx \begin{cases} 1\,, & \beta \ll 1\,, \\ \beta \equiv \gamma\alpha^{-1}\,, & \beta > 1\,. \end{cases} \tag{13}$$

The required τ_{sF}–β dependences derived by the numerical solution of Eq. (1) are presented by the solid curves in Fig. 3. As previously, the dashed line shows the result obtained by speed equation (5), within the adiabatic approximation used. One can see that the presented characteristics that are taken for a wide interval of the slope coefficients α give very close agreement with expression (13). Consequently, the delay time in non-scaled units, τ_{D}, may be evaluated by the following expressions: $\tau_{\mathrm{D}} \approx \gamma^{-1}(\beta - 1)$ if $\beta > 1$, and $\tau_{\mathrm{D}} \approx 0$ if $\beta \ll 1$.

In closing the discussion of τ_{D}–β dependences let us touch briefly on the spurious drift of BF. We have already noted that our previous results derived by the use of speed equation (5) have showed that the square-wave ac driving described by the "superposition" of the step-like forcing functions appeared to be very "effective". As noted, the speed equation (5) neglects the retardation. It is quite obvious that the occurrence of the lag time between the driving force and the propagation velocity of the front will shrink the spurious drift discussed; the "front-ratchet" effect will be suppressed if the retardation is large enough. From Eq. (13) it follows that the lag time τ_{D} increases with the increasing β, namely, one has that $\tau_{\mathrm{D}} \cong \alpha^{-1}$ if $\gamma \gg \alpha \cong \tau_R^{-1}$. Thus, one may expect that the spurious drift generated by the square-wave ac force will decrease with the increasing parameter $\beta_T := (\alpha T)^{-1}$, where by T we

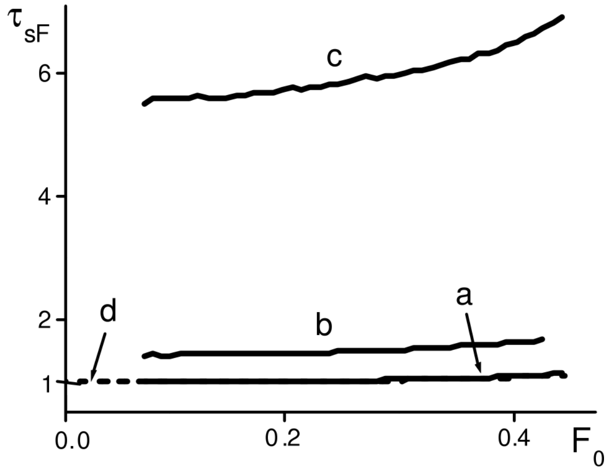

Fig. 4. Dependence of relaxation time τ_{sF} on the strength of driving force. Parameter values are $h_R = 1$, $\alpha = 1.0$; $\beta = 0.2, 1.0, 5.0$ in solid curves a, b, c, respectively; dashed curve d was received by speed equation (5).

denote the period of the force. This implies that the average drift velocity of the unforced transport of BF, v, will decrease with the decreasing period of the driving force and will vanish in the limit $T \to 0$. Indeed, one has that $\tau_{\mathrm{D}} \to \tau_R \cong \alpha^{-1}$ if $T_{\mathrm{F}} \to 0$, hence, it follows that $(\tau_{\mathrm{D}}/T) \to \infty$ if $T \to 0$. This implies that in the high frequency range of the driving ac force the considered retardation, if taken in the relative units of the period T, becomes extremely large. Our preliminary results derived by use of the square-wave forcing function confirm this conclusion: the drift velocity of the spurious drift, v, decreases with the increasing "rate" of the driving force, β_T. The peculiarities of spurious drift generated in the cases of both the slow and the fast driving will be discussed more extensively elsewhere.

Let us turn to τ_{sF}–F_0 characteristics and discuss the dependence of the lag time versus the strength of the driving force $f(t)$. As earlier, we shall use the scaled units; the relaxation time τ_{S} is scaled in the units of the switching time T_{F}. We have already noted that the peculiarities of τ_{sF}–β characteristics do not depend on the slope coefficient α (see Fig. 3). Thus, without loss of generality, we take that $\alpha = 1$. The considered τ_{sF}–F_0 dependences derived by the numerical solution of Eq. (1) are presented by curves a, b, and c in Fig. 4. The dashed curve d shows the result derived by the speed equation (5). One can see that the relaxation time τ_{sF} is a gently sloped function of F_0; the presented characteristics are flattened in the both cases of slow and fast driving. The relative deviation of the parameter τ_{sF} does not exceed few tenths. Further, in the case of slow driving ($\beta = 0.2$) the propagation velocity of BF immediately follows the driving force for any strength F_0; the presented curves a and d that have been derived by use of equations (1) and (5), respectively, practically coincide. Finally, the increase of the parameter β practically does not influence the slope of τ_{sF}–F_0 characteristics; the relaxation time τ_{sF} slightly increases with the increasing magnitude of F_0 in the both cases of slow and fast driving (compare curves a and c). We conclude by noting that the influence of the strength F_0 on the considered lag time is insignificant (within the pseudolinear model). Thus, it seems to be most likely that the decrease of the spurious drift of a rapidly driven BF would come basically through the dependence of retardation on the rate (frequency) of the applied zero-mean force.

4. Conclusions

The retarded accelerations of the self-organized front in a continuous bistable system being under the action of the step-like force have been studied by considering the lag (delay) time between the driving force and the propagation velocity of the driven front. The propagation of the "bistable" front separating two steady states of the bistable system has been described by the nonlinear differential equation of reaction-diffusion type, with the rate function approximated by linear pieces. The basic characteristics of the retarded response, namely, the dependences of the delay time on both the rate (steepness) and the strength (magnitude) of the driving force have been derived in a wide interval of governing parameters of the step-like force, for arbitrary strengths and rates of the force. By tuning the adjustable parameters of both the driving force and the rate function we have found that the delay time depends both on the steepness of the force and on the slope coefficients of piecewise-linear rate function. In contrast, the considered delay time practically does not depend on the strength of the driving force. The derived characteristics show that the considered retardation is small if the characteristic relaxation rates in the system significantly exceed the rate of the driving force. The delay time increases with the increasing rate of the driving force and approaches some fixed value given by the slope coefficients of the rate function when the step-like driving becomes extremely steep. The presented characteristics evidently show that the spurious drift of BF in "front-ratchet" device should be sensitive to the rate of zero-mean force. A decrease of the average velocity of spurious drift is expected with the increasing rate (frequency) of the ac force.

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VĖLUOJANTIS SAVAIMINĖS FRONTINĖS SANDAROS ATSAKAS Į SPARČIAI KINTANČIĄ JĖGĄ

A. Raguotis [a], F. Ivanauskas [a,b], R. Bakanas [c]

[a] *Matematikos ir informatikos institutas, Vilnius, Lietuva*
[b] *Vilniaus universitetas, Vilnius, Lietuva*
[c] *Puslaidininkių fizikos institutas, Vilnius, Lietuva*

Santrauka

Nagrinėjamas bistabilaus fronto (savaiminės frontinės sandaros) atsakas į sparčiai kintančią jėgą. Veikančioji jėga aprašoma laipto pavidalo jėgos funkcija, kurios įjungimo trukmė, nusakanti jėgos kitimo spartą, ir jos amplitudė, atitinkanti jėgos stiprį, yra laisvai parenkami parametrai. Fronto sklidimas aprašomas netiesine reakcijos-difuzijos tipo diferencialine lygtimi, naudojant "pseudotiesinį" bistabilios terpės modelį, kurio spartos funkcija yra visur tiesinė, išskyrus šios funkcijos ekstremumo (lūžio) taškus, kuriuose funkcijos išvestinė yra trūki. Naudojant skaitinius bei analizinius metodus, parodyta, kad fronto atsakas (jo akimirkinio greičio relaksacija) smarkiai vėluoja jėgos atžvilgiu, jeigu jėgos įjungimo trukmė pakankamai maža, o jėgos sparta didelė. Būdinga atsako vėlinimo trukmė, randama palyginimo būdu, – lyginant veikiančios jėgos bei fronto sklidimo greičio relaksacijos spartas, priklauso tiek nuo jėgos įjungimo trukmės, jos augimo greičio, tiek ir nuo spartos funkcijos parametrų, nusakančių "vidinių" relaksacijos vyksmų spartą bistabilioje terpėje. Parodyta, kad vėlinimo trukmė beveik nepriklauso nuo veikiančios jėgos stiprio (amplitudės). Gautieji rezultatai yra svarbūs, aprašant dažnines "front-ratchet'ų" charakteristikas, kurios iki šiol dar nėra tyrinėtos. Tiesioginio veikimo "ratchet"-mechanizmas buvo aprašytas ankstesniuose mūsų darbuose [4], naudojant kvazistatinį lėtai kintančios jėgos artinį. Vėluojančio frontų atsako charakteristikos, pateiktos šiame darbe, rodo, kad kryptingas bistabilaus fronto dreifas "nulinės" jėgos lauke turėtų gerokai sumažėti, esant pakankamai dideliems jėgos dažniams, viršijantiems būdingas relaksacijos trukmes sistemoje.