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PHASE AND GROUP VELOCITIES IN THE MEDIA WITH NON-DEBYE RELAXATION

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Dispersion and absorption in the media with stretch exponential or Kohlrausch–Williams–Watts relaxation are considered. The frequency-dependent dispersion and absorption coefficients, phase and group velocities are obtained and compared with Debye–Mandelshtam–Leontovich expressions.

Keywords: non-Debye relaxation, phase and group velocities

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1. Introduction

One of the reasons for the appearance of acoustic wave dispersion in a medium is the relaxation processes. In this case, the thermodynamic equilibrium in a medium regarding compressions and rarefactions is established with certain delay [1]. The equation of the state of the medium, which relates the pressure p and the density ρ, becomes dependent on density variations in the previous moments and thus nonlocal in time, and is described by the integral equation

$$p(x,t) = c_\infty^2 \rho(x,t) + \int_0^{+\infty} R(t')\rho(x,t-t')\,\mathrm{d}t' . \quad (1)$$

Here we deliberately extract the first term $c_\infty^2\rho$ to avoid the δ-function in the expression for the relaxation function $R(t)$ below.

The explicit form of the relaxation function $R(t)$ depends on the internal structure of the medium and is studied by molecular acoustics and acoustic spectroscopy [2, 3]. However, a series of significant results can be obtained without specifying the physical nature of the interacting forces and solely on the grounds of a phenomenological consideration of the relaxation processes. For instance, the Debye–Mandelshtam–Leontovich (DML) theory of relaxation is based on the assumption that the contribution of the "previous" effects exponentially decreases with increasing the time of their retardation [4, 5]. In the case of a single process of relaxation, this results in the following equation of state of the medium:

$$p(x,t) = c_0^2\rho(x,t) + mc_0^2 \int_{-\infty}^{t} \mathrm{e}^{-(t-t')/\tau} \frac{\partial}{\partial t'}\rho(x,t')\,\mathrm{d}t' , \quad (2)$$

which is equivalent to equation (1) with the relaxation function

$$R(t) = \frac{mc_0^2}{\tau}\,\mathrm{e}^{-t/\tau} , \quad (3)$$

where $m = (c_\infty^2 - c_0^2)/c_0^2$ is the parameter characterizing the relative value of the phase velocity changes of the wave with an increase in its frequency; c_0 and c_∞ are the velocities of the low-frequency ($\omega\tau \to 0$) and high-frequency ($\omega\tau \to \infty$) sound ($c_\infty > c_0$), and τ is a characteristic time of relaxation.

Since the classical works by Debye, Mandelshtam, and Leontovich on the theory of relaxation the experimental data have been accumulating to prove deviations from the classical exponential law of relaxation. The first note on the possible non-exponential character of relaxation by Kohlrausch dates as far back as the mid-19th century, nevertheless, only in the recent decades, due to the new technologies and much higher accuracy of experimental studies, non-exponential relaxation has become an object of comprehensive investigations. This concerns the experiments with both dielectrics [2, 3] and liquids, gases and polymers [6–8]. A number of empirical expressions have been proposed to describe relaxation in these media.

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The present work shows that deviation from classical exponential relaxation leads to frequency dependence changes of the coefficients of absorption and dispersion in such media. Dependences for the related phase and group velocities have been obtained. The results are compared among themselves and with classical exponential dependence.

2. Waves in a medium with stretch exponential relaxation

Experimental studies show [6–8] the presence of a vast class of substances in which relaxation is described by the relaxation function of a more complicated form:

$$R(t) = mc_0^2\,\frac{\alpha}{\tau}\left(\frac{t}{\tau}\right)^{\alpha-1}\mathrm{e}^{(-t/\tau)^\alpha}\,, \tag{4}$$

where the parameter $0 < \alpha < 1$. This results in the equation of state

$$p(x,t) = \tag{5}$$

$$c_0^2\rho(x,t) + mc_0^2\int_{-\infty}^{t}\mathrm{e}^{-[(t-t')/\tau]^\alpha}\,\frac{\partial}{\partial t'}\rho(x,t')\,\mathrm{d}t'\,,$$

which describes the stretch exponential or Kohlrausch–Williams–Watts (KWW) relaxation processes [6, 7].

One-dimensional sound waves of small amplitude in immovable gas are described by a system of linear equations

$$\rho_0\,\frac{\partial V}{\partial t} + \frac{\partial \Delta p}{\partial x} = 0\,, \quad \frac{\partial \Delta\rho}{\partial t} + \rho_0\,\frac{\partial V}{\partial x} = 0\,, \tag{6}$$

where $\Delta\rho, \Delta p$, and V are, respectively, small perturbations of the density, pressure, and velocity of gas or liquid particles regarding their equilibrium values $\rho = \rho_0, p = p_0, V_0 = 0$ [9].

This system of equations is not closed. It should be supplemented by equations of state to interrelate the pressure, density, and other thermodynamic characteristics of the medium. For adiabatic processes in gases, the Poisson adiabate serves as the equation of state:

$$p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{7}$$

where γ is the index of the adiabate $(1 < \gamma < 2)$ [1]. For liquids, as the state equation we can use the Tate equation

$$p(\rho) = A_0\left(\frac{\rho}{\rho_0}\right)^{\bar\gamma} - A_1\,, \tag{8}$$

where $\bar\gamma, A_{0,1}$ are empiric constants [9]. The value $p_0 = A_0 - A_1 \cong \rho_0 c^2$ is of the order of the "internal" pressure in the liquid, which is related to the forces of intermolecular interaction.

Both for gases and liquids, small pressure and density perturbations, as follows from the respective equations of state (7) and (8), are related by a simple correlation

$$\Delta p = c_\mathrm{s}^2\Delta\rho\,, \tag{9}$$

where $c_\mathrm{s} = \sqrt{\gamma p/\rho}$ for the gas and $c_\mathrm{s} = \sqrt{\bar\gamma p/\rho}$ for the liquid are the adiabatic velocities of sound.

As follows from the general nonlocal state equation of the medium (1), independently of the form of the relaxation function $R(t)$ the perturbations of density $\Delta\rho$ and pressure Δp, due to the linear character of equation (1), obey the same equation of state. Substituting the expression for the pressure perturbation Δp, which follows from expression (5), into the system of equations (6) and excluding the variables V and p, we obtain the integro-differential equation for the perturbation of density:

$$\frac{1}{c_0^2}\,\frac{\partial^2\Delta\rho}{\partial t^2} - \frac{\partial^2\Delta\rho}{\partial x^2} = \tag{10}$$

$$m\frac{\partial^2}{\partial x^2}\left[\int_{-\infty}^{t}\mathrm{e}^{-[(t-t')/\tau]^\alpha}\,\frac{\partial}{\partial t'}\Delta\rho(x,t')\,\mathrm{d}t'\right].$$

3. Dispersion and absorption in medium with stretch exponential relaxation

As follows from equation (10), the dispersion equation interrelating the frequency ω and the wave number k in a medium with KWW relaxation is

$$k^2\left[1 + mI^\alpha(\omega\tau)\right] = \frac{\omega^2}{c_0^2}\,, \tag{11}$$

where the complex value $I^\alpha(\omega\tau)$ is determined by the integral

$$I^\alpha(x) = \mathrm{i}\int_0^{+\infty}\mathrm{e}^{-(\xi/x)^\alpha - \mathrm{i}\xi}\,\mathrm{d}\xi\,. \tag{12}$$

Usually the parameter $m = (c_\infty^2 - c_0^2)/c_0^2 \simeq 2(c_\infty - c_0)/c_0$ characterizing the relative value of the change of phase velocity with an increase of its density is small [9]. Expanding the expression (11) for $k(\omega) = k' + \mathrm{i}k''$ into a series by the small parameter m we obtain

$$k \simeq \pm\frac{\omega}{c_0}\left[1 - \frac{m}{2}I^\alpha(\omega\tau)\right] = \tag{13}$$

$$\pm\frac{\omega}{c_0}\left\{1 - \frac{m}{2}F^\alpha[\sin(\omega\tau)] + \mathrm{i}\frac{m}{2}F^\alpha[\cos(\omega\tau)]\right\},$$

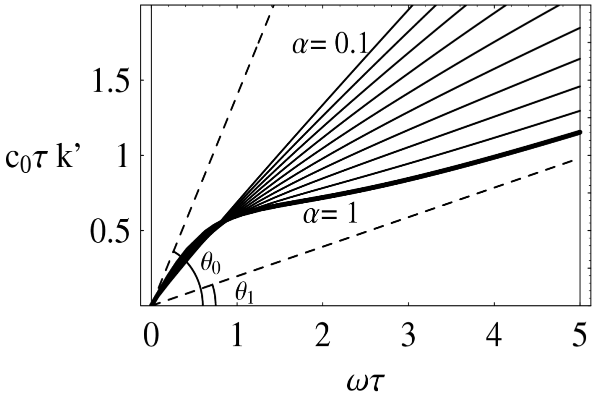

Fig. 1. The dispersion dependence $\omega = \omega(k)$: $c_0\tau k' = (\omega\tau)\left\{1-(m/2)\int_0^\infty \exp\left[-(\xi/\omega\tau)^\alpha\right]\sin\xi\,\mathrm{d}\xi\right\}$ for a medium with KWW relaxation in dimensionless variables $c_0\tau k'$ and $\omega\tau$ for different values of the nonlocality parameter $\alpha = 0.1,\dots,0.9$. The value of the relative magnitude of phase velocity m for the sake of obviousness is inflated up to $m/2 = 0.8$. For $\alpha = 1.0$ we obtain dispersion caused by classical DML relaxation.

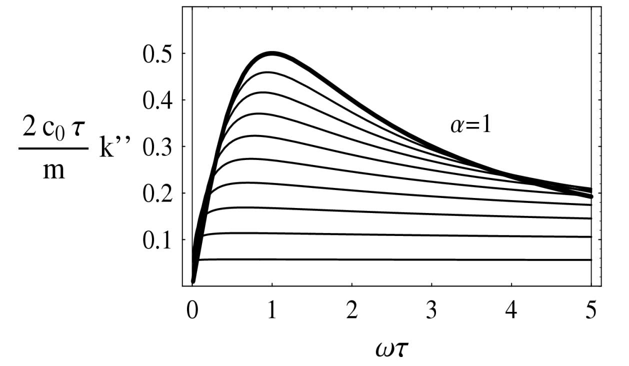

Fig. 2. Dependence of the coefficient of absorption $k'' = (m/2c_0\tau)\int_0^\infty \exp\left[-(\xi/\omega\tau)^\alpha\right]\cos\xi\,\mathrm{d}\xi$ in dimensionless variables $2c_0\tau/m$ and $\omega\tau$ for a medium with KWW relaxation. The value of the relative magnitude of phase velocity m for the sake of obviousness is $m/2 = 0.8$, and different values of the nonlocality parameter $\alpha = 0.1,\dots,0.9$. For $\alpha = 1.0$ we obtain dispersion caused by classical DML relaxation.

where

$$F^\alpha[f(x)] = \int_0^\infty \mathrm{e}^{-(\xi/x)^\alpha} f(\xi)\,\mathrm{d}\xi\,. \qquad (14)$$

From the expression (13) it follows that the propagation of sound wave in a medium with KWW relaxation is always accompanied by absorption and concomitant dispersion:

$$k'' = \frac{m\omega}{2c_0}\,F^\alpha[\cos(\omega\tau)]\,,$$

$$k' = \frac{\omega}{c_0}\left\{1-\frac{m}{2}\,F^\alpha[\sin(\omega\tau)]\right\}. \qquad (15)$$

The dispersion curve $\omega = \omega(k)$ is shown in Fig. 1. From the dependence $\omega(k)$ one can see that the presence of exponential relaxation involves an increase of the velocity of sound with an increase of frequency. Thus, dispersion in a medium with KWW relaxation is anomalous.

The asymptotic values of the dispersion curve are straight lines with an angle of incidence to the x axis, $\theta = \arctan(1-m/2\,F^\alpha[\sin x])$ at $x\to+\infty$. Because of the properties of the function $F^\alpha[g(x)]$ the incidence angle of the asymptote varies from $\theta_0 = \pi/4$ to $\theta_1 = \arctan(1-m/2)$. For $\alpha\to 1$, the dispersion curve for a medium with KWW relaxation turns into a dispersion curve for a medium with DML relaxation:

$$k' = \frac{\omega}{c_0}\left[1-\frac{m}{2}\,\frac{(\omega\tau)^2}{1+(\omega\tau)^2}\right]. \qquad (16)$$

For $\alpha\to 0$ we get a linear interrelation $\omega = c_0 k$ for a harmonic wave.

Figure 2 shows the dependence of the coefficient of absorption $k''(\omega\tau)$ on the dimensionless variable $\omega\tau$. When $\alpha\to 0$, the coefficient of absorption $k''(\omega\tau)\to 0$ not only for $\omega\tau\to\infty$, but also for the maximum value of $k''_{\max}$ determined from the condition $F[\langle\xi^\alpha\cos\xi\rangle] = 0$. For $\alpha\to 1$, the coefficient of absorption $k''(\omega\tau)$ for a medium with KWW relaxation turns into the coefficient of absorption $k''(\omega\tau)$ for a medium with DML relaxation:

$$k'' = \frac{m}{2c_0}\,\frac{\omega^2\tau}{1+(\omega\tau)^2}\,. \qquad (17)$$

For this relaxation, the maximum value of $k'' = k''_{\max} = m\omega/(4c_0)$ is obtained when $\omega\tau = 1$.

Figure 3 presents the phase V_{ph} and group V_{gr} velocities in a medium with KWW relaxation in dependence on the dimensionless parameter $\omega\tau$:

$$V_{\mathrm{ph}} = \frac{\omega}{\mathrm{Re}\,k} = c_0\left\{1+\frac{m}{2}F^\alpha[\sin(\omega\tau)]\right\}, \qquad (18)$$

$$V_{\mathrm{gr}} = \frac{d\omega}{dk} =$$

$$c_0\left\{1+\frac{m}{2}F^\alpha[\sin(\omega\tau)]\right. \qquad (19)$$

$$\left.-\frac{\alpha m}{2(\omega\tau)^\alpha}F^\alpha[(\omega\tau)^\alpha\sin(\omega\tau)]\right\}.$$

The above dependence shows that the phase velocity

$$c_0 < V_{\mathrm{ph}} < c_0\left(1+\frac{m}{2}F^\alpha[\sin(\omega\tau)]\right) \qquad (20)$$

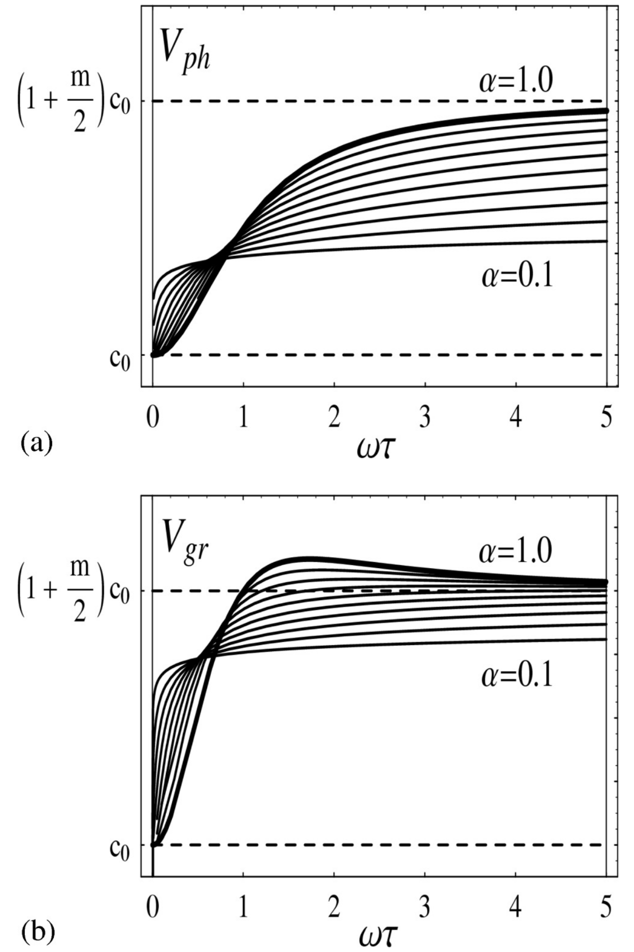

Fig. 3. Dependence of (a) phase, V_{ph}, and (b) group, V_{gr}, velocities (18), (19) in a medium with KWW relaxation on the dimensionless parameter $\omega\tau$. The solid line corresponds to the phase V_{ph} and group V_{gr} velocities in a medium with DML relaxation.

for $\omega\tau \to +\infty$, and the maximum value is reached when $\alpha \to 1$, i. e. in the case of DML relaxation

$$V_{\mathrm{ph}}^{\mathrm{DML}} = c_0 \left[1 + \frac{m}{2} \frac{(\omega\tau)^2}{1+(\omega\tau)^2} \right] . \tag{21}$$

The group velocity V_{gr} in the region of high frequencies $\omega\tau \gg 1$ always exceeds c_0. For $\alpha \to 0$, $V_{\mathrm{gr}} \to c_0$ in all the frequency range, and when $\alpha \to 1$ it turns into a group velocity for a medium with DML relaxation:

$$V_{\mathrm{gr}}^{\mathrm{DML}} = c_0 \left\{ 1 + \frac{m}{2} \frac{(\omega\tau)^2[3+(\omega\tau)^2]}{[1+(\omega\tau)^2]^2} \right\} . \tag{22}$$

Note here that there is a critical value of the quantity $\alpha_{\mathrm{cr}} \sim 0.56$, which is obtained from the condition $V_{\mathrm{gr\,max}} \le (1+m/2)c_0$. For $\alpha \le \alpha_{\mathrm{cr}}$ we have no maximum for the group velocity V_{gr}, and the qualitative dependences of the phase and group velocities coincide.

4. Conclusions and discussion

The relaxation function $R(t)$ considered above certainly does not exhaust all the possible kinds of relaxation processes. For instance, in dielectric media three types of relaxation functions are very popular: Cole–Cole, Cole–Davidson, and Hawriliak–Negami [10]. Without any doubt such an approach could be applied to other types of the relaxation function. The formula for the state equation (1) may be presented as $p = c_\infty^2[1+\hat{R}]\rho$, where the expression in square brackets may be interpreted as the operator relation. Such approach allows to see analogies with solid body deformation, in which for some simplest cases the tension–deformation correlation $\sigma = E\varepsilon$ is substituted by the expression $\sigma = \hat{E}\varepsilon$.

The relaxation damping of perturbation propagating in a medium with DML relaxation is observed both in gases and liquids, e. g., in sea water. Of importance is the fact that the different content of ionic impurities can result in a deviation of the KWW relaxation type, in which the damping, as follows from (17), for $\alpha \to 0$ is more rapid.

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FAZINIS IR GRUPINIS GREIČIAI NE DEBAJAUS RELAKSACIJOS APLINKOJE

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Santrauka

Pateikiamos fazinio ir grupinio greičių išraiškos ne Debajaus relaksacijos aplinkoje. Parodyta, kad Kohlrausch, Williams ir Watts relaksacijos atveju dažninės fazinio ir grupinio greičių priklausomybės kokybiškai skiriasi nuo analogiškų priklausomybių Debajaus relaksacijos atveju. Įvertinti dispersijos ir sugerties koeficientai. Šitokiu metodu gali būti nagrinėjami dispersiniai sąryšiai ir kitose aplinkose su kitomis relaksacijos funkcijomis.