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Phase reduction of weakly perturbed limit cycle oscillations in time-delay systems

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ABSTRACT

The phase reduction method is applied to a general class of weakly perturbed time-delay systems exhibiting periodic oscillations. The adjoint equation with an appropriate initial condition for the infinitesimal phase response curve of a time-delay system is derived. The method is demonstrated numerically for the Mackey–Glass equation as well as for a chaotic Rössler system subject to a delayed feedback control (DFC). We show that the profile of the phase response curve of a periodic orbit stabilized by the DFC algorithm does not depend on the control matrix. This property is universal and holds for any dynamical system subject to the DFC.

1. Introduction

Time-delay effects are very common in many natural and engineering systems. They may arise due to the finite speed of propagation of information between the constituents of a complex system or due to processing and latency times. Systems with time delay have been widely studied in fields as diverse as biology [1,2], population dynamics [3], neural networks [4], feedback controlled mechanical systems [5], machine tool vibrations [6], lasers [7], etc (cf. Ref. [8] for a review). Time-delay feedback has been shown to represent an efficient tool for the noninvasive control of unstable periodic orbits or steady states of chaotic systems [9–11].

Time-delay systems are modeled by delay differential equations (DDEs). Generally, the DDEs are more complicated than the ordinary differential equations (ODEs), since their phase spaces are infinite-dimensional. However, under certain conditions, the phase space of a time-delay system can be reduced, and its dynamics can be represented by a simple system of ODEs. Such a situation appears, e.g., at a Hopf bifurcation, when the time-delay dynamics reduces to a normal form on a surface of the center manifold, which describes the birth of a small limit cycle (e.g., [12]). Another situation, when a time-delay system admits a description using reduced equations, can appear far away from a bifurcation point. If a time-delay system has a stable limit cycle (it can be far away from the bifurcation point) and is disturbed by a small time-dependent perturbation, then the equations of the system can be reduced to a scalar phase equation. The technique allowing such a reduction is known as a phase reduction method (e.g., [13,14]). An inherent characteristic of the limit cycle oscillator resulting from the phase reduction procedure is the infinitesimal phase response (or phase resetting) curve (PRC) [14,15]. The PRC describes the dependence of the phase shift of the oscillation in response to a small perturbing pulse at each phase of the oscillation. The study and applications of the PRCs are currently receiving growing attention as regards their theoretical and experimental aspects [16,17]. Investigation of the PRCs is relevant for understanding the interaction properties of the neural networks, such as their stability [18] or synchronization and clustering [19]. While mostly studied in the domain of neurons [16,20,21,15,22,18,23,24], the PRCs are also explored in other oscillatory systems, such as cardiac systems [25], coupled circadian clocks of insects [26], a periodically driven saline oscillator [27], etc.

Although any weakly perturbed rhythmic system can be reduced to a phase model, most investigations in the field of phase reduction are devoted to the systems described by the ODEs. To the best of our knowledge, the phase reduction method has not been applied to oscillatory systems possessing inherent delays. Note that oscillatory systems with time-delay couplings have been extensively studied in the context of the Kuramoto type phase models [28–33]. Such couplings mimic the finite speed of propagation of signals between oscillatory units, e.g., along axons in a neural network. However, in these models, the oscillatory units were assumed to be without inherent delays. The delay terms that describe the couplings were considered as small perturbations to the original oscillators. The reduced phase equations in this case

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are infinite-dimensional, since the couplings in these equations are still represented by time-delay terms.

In this paper, we systematically develop the phase reduction procedure for a general class of time-delay systems exhibiting periodic limit cycle oscillations. In our consideration, we suppose that time-delay effects are essential for the formation of the limit cycle and the PRC of the system. In contrast to the case for previous investigations, the time-delay terms are not regarded as small perturbations and the derivation of the PRC is based on the consideration of the system dynamics in an infinite-dimensional phase space. The reduced phase equation in our case represents a simple scalar equation that does not contain time-delay terms. We present two methods of derivation of the phase reduced equation for time-delay systems. The first ("heuristic") method is based on physical arguments and uses the representation of the delay term by a delay line, which we model by an advection equation. By discretizing the space variable of the advection equation, we come to a finite set of ODEs to which we apply the classical PRC theory. Then we return to a continuous limit and obtain the PRC for the original DDE. The second (direct) method deals directly with the DDEs without recourse to the ODEs; it is mathematically more rigorous, but less obvious from a physical standpoint. We show that the reduced phase equation for DDE systems is essentially the same as that for ODE systems; however, the PRC is defined by a difference–differential equation of the advanced type. The latter can be solved by a backward integration. Although the equation for the PRC still represents an infinite-dimensional problem, we need to solve this equation only once to obtain the PRC for a given time-delay system. Then knowing the PRC, the response of the system to any time-dependent perturbation can be easily determined from the phase equation that represents a simple scalar ODE. The rest of the paper is organized as follows. In Section 2, we present the two methods of derivation of the phase reduced equations for DDE systems. In Section 3, the general theoretical results are demonstrated numerically for the Mackey–Glass equation [2]. Here we show the efficiency of the phase reduction procedure for estimating the Arnold tongues. Section 4 is devoted to the application of the phase reduction method to chaotic systems subject to the DFC and a small external perturbation. We reveal the interesting general property that the profile of the PRC of a periodic orbit stabilized by the DFC is invariant under any variation of the control matrix. The concluding part is presented in Section 5.

2. Phase reduction of time-delay systems

2.1. Phase reduction of time-delay systems via approximation of the DDE using ODEs

In this subsection, we present a "heuristic" derivation of phase reduced equations for the DDE system. We first recall the results of the classical phase reduction theory for the ODEs. To extend these results to a DDE system we rewrite the DDE as an ODE coupled with an advection equation. The advection equation is introduced in order to model the time-delay feedback in the DDE. By discretizing the space variable of the advection equation, we transform the DDE to a finite set of ODEs. For this system, we apply the results of the classical phase reduction theory and then returning to the continuous limit, we derive the phase reduced equations for the original DDE. An alternative method of derivation of the phase reduced equations directly from the DDE system is presented in Section 2.2.

2.1.1. Phase reduction theory for ODEs

The classical phase reduction theory is usually formulated for a weakly perturbed limit cycle oscillator described by the ODEs of the form

$$\dot{\mathbf{y}}(t) = \mathbf{G}(\mathbf{y}(t)) + \varepsilon\boldsymbol{\phi}(t). \tag{1}$$

Here $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ is an n-dimensional state variable (column vector) of the system and $\varepsilon\boldsymbol{\phi}(t) = \varepsilon(\phi_1(t), \phi_2(t), \ldots, \phi_n(t))^T$ represents a small time-dependent perturbation, where $\varepsilon \ll 1$ is a small parameter. We suppose that for $\varepsilon = 0$ the system has a stable limit cycle $\tilde{\mathbf{y}}(t)$ with a period T: $\tilde{\mathbf{y}}(t) = \tilde{\mathbf{y}}(t+T)$. For $\varepsilon = 0$, the oscillation phase $\varphi = \varphi(\mathbf{y})$ can be introduced on the limit cycle and within its finite vicinity in such a way that $\dot{\varphi}(t) = 1$ (cf. [14]). The limit cycle can be parametrized with the phase, $\tilde{\mathbf{y}}(\varphi) = \tilde{\mathbf{y}}(\varphi + T)$. In the presence of a weak forcing the phase description can be still utilized [14]: $\dot{\varphi}(t) = 1 + \varepsilon\sum_{i=1}^{n}(\partial\varphi(\mathbf{y})/\partial y_i)_{\mathbf{y}=\tilde{\mathbf{y}}(\varphi(t))}\phi_i(t)$. In a more usual form this equation reads:

$$\dot{\varphi}(t) = 1 + \varepsilon\mathbf{z}^T(\varphi(t))\boldsymbol{\phi}(t) + o(\varepsilon). \tag{2}$$

Here $\mathbf{z}^T$ is the transpose vector of $\mathbf{z}$, where $\mathbf{z} = (z_1, z_2, \ldots, z_n)^T$ is an n-dimensional T-periodic vector function $\mathbf{z}(\varphi + T) = \mathbf{z}(\varphi)$ referred to as the infinitesimal PRC and $o(\varepsilon)$ denotes the error terms smaller than ε such that $o(\varepsilon)/\varepsilon \to 0$ if $\varepsilon \to 0$ [20]. The PRC represents the gradient of the phase estimated on the limit cycle $z_i = (\partial\varphi(\mathbf{y})/\partial y_i)_{\mathbf{y}=\tilde{\mathbf{y}}(\varphi)}$ and practically can be computed as a T-periodic solution of the adjoint equation (see, e.g., [20])

$$\dot{\mathbf{z}}(t) = -[D\mathbf{G}(\mathbf{y})]^T_{\mathbf{y}=\tilde{\mathbf{y}}(t)}\mathbf{z}(t) \tag{3}$$

with an initial condition

$$\mathbf{z}^T(0)\dot{\tilde{\mathbf{y}}}(0) = 1. \tag{4}$$

Here $[D\mathbf{G}(\mathbf{y})]_{\mathbf{y}=\tilde{\mathbf{y}}(t)}$ is the Jacobian of the unperturbed system (1) evaluated on the limit cycle. Eqs. (2)–(4) represent the main results of the phase reduction theory for the ODEs. Below we shall use these results to derive similar equations for the DDEs. Note that Eq. (3) is unstable and its numerical solution is usually obtained via a backward integration [15]. Since the Jacobian $[D\mathbf{G}(\mathbf{y})]_{\mathbf{y}=\tilde{\mathbf{y}}(t)}$ is usually unavailable in an analytical form, its values are estimated from a forward integration of the unperturbed system (1).

2.1.2. Approximation of the DDE system by using ODEs

Now consider a weakly perturbed limit cycle oscillator described by a system of DDEs:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{x}(t-\tau)) + \varepsilon\boldsymbol{\psi}(t). \tag{5}$$

Here $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is an n-dimensional vector, τ is a delay time, and $\varepsilon\boldsymbol{\psi}(t) = \varepsilon(\psi_1(t), \psi_2(t), \ldots, \psi_n(t))^T$ represents a small time-dependent perturbation, where $\varepsilon \ll 1$ is a small parameter. We suppose that for $\varepsilon = 0$ the system has a stable limit cycle $\tilde{\mathbf{x}}(t)$ with a period T: $\tilde{\mathbf{x}}(t) = \tilde{\mathbf{x}}(t+T)$.

Physically, the time-delay feedback in system (5) can be implemented via a delay line, which can be modeled by an advection equation. Thus Eq. (5) can be rewritten in a mathematically equivalent form as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \boldsymbol{\xi}(\tau, t)) + \varepsilon\boldsymbol{\psi}(t), \tag{6a}$$

$$\frac{\partial\boldsymbol{\xi}(s,t)}{\partial t} = -\frac{\partial\boldsymbol{\xi}(s,t)}{\partial s}, \qquad \boldsymbol{\xi}(0,t) = \mathbf{x}(t), \tag{6b}$$

where $\boldsymbol{\xi}$ is a vector variable of the advection equation and $s \in [0, \tau]$ is a space variable. We take the delay line of the length τ and the velocity of the wave equal to unity such that the signal at the input of the delay line $\boldsymbol{\xi}(0,t) = \mathbf{x}(t)$ is delayed at the output by the amount τ: $\boldsymbol{\xi}(\tau, t) = \mathbf{x}(t-\tau)$.

Now we discretize the space variable of the advection equation by dividing it into N equal intervals $s_i = i\tau/N$, $i = 0, \ldots, N$, and approximate the space derivative of Eq. (6b) by a finite difference $[\partial\boldsymbol{\xi}(s,t)/\partial s]_{s=s_i} \approx [\boldsymbol{\xi}(s_i,t) - \boldsymbol{\xi}(s_{i-1},t)]N/\tau$. Defining $\mathbf{x}_0(t) = \mathbf{x}(t)$

and $\mathbf{x}_i(t) = \boldsymbol{\xi}(i\tau/N, t) \approx \mathbf{x}(t - i\tau/N)$, $i = 1, \ldots, N$, we get a system of $n \times (N+1)$ ODEs

$$\dot{\mathbf{x}}_0(t) = \mathbf{F}(\mathbf{x}_0(t), \mathbf{x}_N(t)) + \varepsilon\boldsymbol{\psi}(t), \tag{7a}$$

$$\dot{\mathbf{x}}_1(t) = [\mathbf{x}_0(t) - \mathbf{x}_1(t)]N/\tau, \tag{7b}$$

$$\vdots$$

$$\dot{\mathbf{x}}_N(t) = [\mathbf{x}_{N-1}(t) - \mathbf{x}_N(t)]N/\tau, \tag{7c}$$

which approximate Eqs. (6) as well as the time-delay system (5). For $N \to \infty$, the system of Eqs. (7) transforms to Eqs. (6) and thus its solution approaches the solution of the time-delay system (5), $\mathbf{x}_0(t) \to \mathbf{x}(t)$. We emphasize that for any finite N, Eqs. (7) represent the finite ODE system, and we can utilize the results of phase reduction theory presented in the previous subsection. The system (7) can be rewritten in the form of Eq. (1) by using the notation

$$\mathbf{y}(t) = \begin{pmatrix} \mathbf{x}_0(t) \\ \mathbf{x}_1(t) \\ \vdots \\ \mathbf{x}_N(t) \end{pmatrix}, \qquad \mathbf{G}(t) = \begin{pmatrix} \mathbf{F}(\mathbf{x}_0(t), \mathbf{x}_N(t)) \\ [\mathbf{x}_0(t) - \mathbf{x}_1(t)]N/\tau \\ \vdots \\ [\mathbf{x}_{N-1}(t) - \mathbf{x}_N(t)]N/\tau \end{pmatrix},$$
$$\boldsymbol{\phi}(t) = \begin{pmatrix} \boldsymbol{\psi}(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{8}$$

In this notation, the Jacobian of the unperturbed system (7) reads

$$D\mathbf{G}(t) = \begin{pmatrix} \mathbf{A}(t) & 0 & 0 & \cdots & \mathbf{B}(t) \\ N/\tau & -N/\tau & 0 & \cdots & 0 \\ 0 & N/\tau & -N/\tau & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -N/\tau \end{pmatrix}, \tag{9}$$

where $\mathbf{A}(t) = [D_1\mathbf{F}(\mathbf{x}_0, \mathbf{x}_N)]_{\mathbf{x}=\tilde{\mathbf{x}}(t)}$ and $\mathbf{B}(t) = [D_2\mathbf{F}(\mathbf{x}_0, \mathbf{x}_N)]_{\mathbf{x}=\tilde{\mathbf{x}}(t)}$ are T-periodic $n \times n$ Jacobian matrices estimated on the limit cycle of the system. The symbols D_1 and D_2 denote the vector derivatives of the function $\mathbf{F}$ with respect to the first and second argument, respectively. The adjoint equation (3) for this system takes the form

$$\begin{pmatrix} \dot{\mathbf{z}}_0(t) \\ \dot{\mathbf{z}}_1(t) \\ \dot{\mathbf{z}}_2(t) \\ \vdots \\ \dot{\mathbf{z}}_N(t) \end{pmatrix} = \begin{pmatrix} -\mathbf{A}^T(t) & -N/\tau & 0 & \cdots & 0 \\ 0 & N/\tau & -N/\tau & \cdots & 0 \\ 0 & 0 & N/\tau & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\mathbf{B}^T(t) & 0 & 0 & \cdots & N/\tau \end{pmatrix} \times \begin{pmatrix} \mathbf{z}_0(t) \\ \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \\ \vdots \\ \mathbf{z}_N(t) \end{pmatrix}, \tag{10}$$

and the initial condition (4) reads

$$\sum_{i=0}^{N} \mathbf{z}_i^T(0)\dot{\tilde{\mathbf{x}}}_i(0) = 1. \tag{11}$$

Since the perturbation in system (7) is applied only to the first expression, Eq. (7a), the function $\boldsymbol{\phi}$ in Eq. (2) has only the first nonzero component (cf. Eq. (8)). Thus the phase Eq. (2) for the system (7) transforms to

$$\dot{\varphi}(t) = 1 + \varepsilon\mathbf{z}_0^T(\varphi(t))\boldsymbol{\psi}(t). \tag{12}$$

Eqs. (10)–(12) represent the phase reduced description for the system (7) that approximates the solution of the DDE (5). To derive the exact phase reduced equations for (5) we have to take the limit $N \to \infty$ in Eqs. (10)–(12).

2.1.3. Phase reduced equations for the DDE system

Now our aim is to transform Eq. (10) to the form of a difference differential equation for large N. This is a problem inverse to that of deriving Eq. (7) from Eq. (5). Since the system (10) is similar to Eqs. (7) we guess that this transformation can be achieved by the following substitutions: $\mathbf{z}_0(t) = \mathbf{z}(t)$ and

$$\mathbf{z}_i(t) = \frac{\tau}{N}\mathbf{B}^T\left(t + \tau - \frac{(i-1)\tau}{N}\right)\mathbf{z}\left(t + \tau - \frac{(i-1)\tau}{N}\right),$$
$$i = 1, \ldots, N. \tag{13}$$

Inserting these expressions into Eq. (10), for $\mathbf{z}(t)$ we derive (in the limit $N \to \infty$) a difference–differential equation of advanced type:

$$\dot{\mathbf{z}}^T(t) = -\mathbf{z}^T(t)\mathbf{A}(t) - \mathbf{z}^T(t+\tau)\mathbf{B}(t+\tau), \tag{14}$$

where $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are T-periodic Jacobian matrices defined as the vector derivatives of the function $\mathbf{F}$ with respect to the first (D_1) and second (D_2) argument, estimated on the limit cycle of the unperturbed system (5):

$$\mathbf{A}(t) = D_1\mathbf{F}(\tilde{\mathbf{x}}(t), \tilde{\mathbf{x}}(t-\tau)), \tag{15a}$$

$$\mathbf{B}(t) = D_2\mathbf{F}(\tilde{\mathbf{x}}(t), \tilde{\mathbf{x}}(t-\tau)). \tag{15b}$$

The initial condition for Eq. (14) is obtained from Eq. (11) by taking the limit $N \to \infty$:

$$\mathbf{z}^T(0)\dot{\tilde{\mathbf{x}}}(0) + \int_{-\tau}^{0} \mathbf{z}^T(\tau+\vartheta)\mathbf{B}(\tau+\vartheta)\dot{\tilde{\mathbf{x}}}(\vartheta)d\vartheta = 1. \tag{16}$$

Finally, the phase equation for the DDE system (5) derived from Eq. (12) in the limit $N \to \infty$ takes the form

$$\dot{\varphi}(t) = 1 + \varepsilon\mathbf{z}^T(\varphi(t))\boldsymbol{\psi}(t). \tag{17}$$

Eqs. (14)–(17) form the complete system for a phase reduced description of weakly perturbed limit cycle oscillations defined by the time-delay system (5). Note that the original problem formulated by the DDE (5) is defined in an infinite-dimensional phase space, while here we have reduced this problem to a single Eq. (17) for the scalar phase variable φ. The perturbed phase dynamics is completely determined by the PRC $\mathbf{z}(\varphi)$ that satisfies the adjoint equation (14) with the initial condition (16). In Section 2.2, the same equations are derived more rigorously, without using any approximation of the DDE with the ODEs. We apply the phase reduction procedure directly to the DDE system and do not appeal to the known theoretical results from the ODE systems.

In Sections 3 and 4, the above theoretical results are demonstrated for specific time-delay systems. For numerical integration of DDEs, we use the `dde23` solver in MATLAB. Note that the adjoint equation (14) defining the PRC of the DDE system is unstable, as is the similar form, Eq. (3), for the ODE systems. To obtain a T-periodic solution of Eq. (14) we use a technique similar to that developed for the ODE systems, namely, we integrate Eq. (14) backward in time. Since the `dde23` solver does not allow the backward-in-time integration we rewrite Eq. (14) in the reversed time variables. We define $\bar{\mathbf{z}}(t) = \mathbf{z}(-t)$, $\bar{\mathbf{A}}(t) = \mathbf{A}(-t)$ and $\bar{\mathbf{B}}(t) = \mathbf{B}(-t)$. Then the adjoint equation (14) transforms to the usual DDE

$$\dot{\bar{\mathbf{z}}}^T(t) = \bar{\mathbf{z}}^T(t)\bar{\mathbf{A}}(t) + \bar{\mathbf{z}}^T(t-\tau)\bar{\mathbf{B}}(t-\tau) \tag{18}$$

that can be analyzed via the `dde23` solver by a forward integration. Since this equation is linear we can start from an arbitrary initial condition and integrate it until a convergence to a periodic solution is attained. Then this solution has to be renormalized in such a way as to satisfy the initial condition (16). Note that the backward integration of the adjoint equation requires a knowledge of the T-periodic matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$. These matrices have to be estimated beforehand by the forward integration of the unperturbed system (5).

2.2. Direct derivation of phase reduced equations for the DDE system

In this subsection, we present a direct method of derivation of the phase reduced equations for the time-delay system (5). Unlike in Section 2.1, here we do not use any approximation of the DDE with ODEs and do not appeal to the known theoretical results from ODE systems.

2.2.1. Phases and isochrons of a limit cycle of the unperturbed DDE system

First, we consider the unperturbed time-delay system (5) for $\varepsilon = 0$:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{x}(t-\tau)), \tag{19}$$

where $\mathbf{x}(t)$ is a vector in an n-dimensional state space Γ. If we are interested in solutions of this equation at times $t \geq 0$, it is necessary to define the state vector $\mathbf{x}(t)$ in the entire interval $[-\tau, 0]$:

$$\mathbf{x}(\vartheta) = \boldsymbol{\chi}(\vartheta), \quad -\tau \leq \vartheta \leq 0, \tag{20}$$

where $\boldsymbol{\chi}(\vartheta)$ is a given continuous vector-valued initial function in a function space $\mathcal{C}$. Here $\mathcal{C}$ is the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$. To ensure the uniqueness of solutions of Eqs. (19) and (20), we cannot restrict our attention to the state space Γ. The state of the time-delay system at time t has to be described by an extended state vector $\mathbf{x}_t(\vartheta) \in \mathcal{C}$ constructed in the interval $[t-\tau, t]$ according to the prescription (e.g., [34])

$$\mathbf{x}_t(\vartheta) = \mathbf{x}(t+\vartheta), \quad -\tau \leq \vartheta \leq 0. \tag{21}$$

We suppose that the system (19) has a stable limit cycle solution $\tilde{\mathbf{x}}(t) = \tilde{\mathbf{x}}(t+T)$. Let us define the initial state on the limit cycle in the function space $\mathcal{C}$ as $\tilde{\boldsymbol{\chi}}^{(0)}(\vartheta) = \tilde{\mathbf{x}}_0(\vartheta)$ and assign for this state the phase $\varphi = 0$. Then we define the phases of other states $\tilde{\mathbf{x}}_t(\vartheta)$ on the limit cycle as $\varphi(\tilde{\mathbf{x}}_t(\vartheta)) = t(\bmod T)$. The phase varies in the interval $[0, T]$ and satisfies a simple differential equation, $\dot{\varphi} = 1$.

Our aim is to extend the definition of the phase for the states outside of the limit cycle. In particular, we will need this definition for the states close to the limit cycle $\mathbf{x}_t(\vartheta) = \tilde{\mathbf{x}}_t(\vartheta) + \delta\mathbf{x}_t(\vartheta)$, where the dynamics of the deviations $\delta\mathbf{x}_t(\vartheta) = \delta\mathbf{x}(t+\vartheta)$ can be described by a linearized equation

$$\delta\dot{\mathbf{x}}(t) = \mathbf{A}(t)\delta\mathbf{x}(t) + \mathbf{B}(t-\tau)\delta\mathbf{x}(t-\tau). \tag{22}$$

Here $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are T-periodic Jacobian matrices defined by Eqs. (15). In the following, we will need an equation adjoint to Eq. (22). Such an equation is defined for a vector row $\mathbf{z}^T(t)$ as follows (cf. [35], p. 359):

$$\dot{\mathbf{z}}^T(t) = -\mathbf{z}^T(t)\mathbf{A}(t) - \mathbf{z}^T(t+\tau)\mathbf{B}(t+\tau). \tag{23}$$

The main feature of the adjoint vector $\mathbf{z}(t)$ satisfying Eq. (23) is that a bilinear form defined as

$$\begin{aligned}(\mathbf{z}(t), \delta\mathbf{x}(t)) &= \mathbf{z}^T(t)\delta\mathbf{x}(t) \\ &+ \int_{-\tau}^{0} \mathbf{z}^T(t+\tau+\vartheta)\mathbf{B}(t+\tau+\vartheta)\delta\mathbf{x}(t+\vartheta)d\vartheta\end{aligned} \tag{24}$$

holds in time, i.e. $(\mathbf{z}(t), \delta\mathbf{x}(t)) = const$. Indeed, by direct differentiation of Eq. (24) it is easy to show that its derivative is equal to zero [35].

We suppose that for the periodic matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ all solutions of Eq. (22) can be decomposed in terms of the Floquet modes. Note that examples exist where this does not hold when τ is an integer multiple of T [36], but these specific cases are beyond the scope of our paper. The equation for the Floquet modes is obtained through the substitution

$$\delta\mathbf{x}(t) = \exp(\lambda t)\mathbf{u}(t) \tag{25}$$

and reads

$$\dot{\mathbf{u}}(t) + \lambda\mathbf{u}(t) = \mathbf{A}(t)\mathbf{u}(t) + \mathbf{B}(t-\tau)e^{-\lambda\tau}\mathbf{u}(t-\tau). \tag{26}$$

Here $\mathbf{u}(t) = \mathbf{u}(t+T)$ is a T-periodic Floquet function and λ is a Floquet exponent. Eq. (26) has an infinite number of linear independent periodic solutions $\mathbf{u}_i(t)$ with different exponents λ_i. One of the exponents is zero: $\lambda_0 = 0$; it describes the variations along the limit cycle. The Floquet function of this exponent can be chosen equal to the derivative of the limit cycle, $\mathbf{u}_0(t) = \dot{\tilde{\mathbf{x}}}(t)$. It is easy to verify that this function satisfies Eq. (26) for $\lambda = 0$. The real parts of the remaining Floquet exponents are negative: $\Re(\lambda_i) < 0$, $i = 1, \ldots, \infty$, because we suppose that the limit cycle is stable.

Substituting the Floquet solutions $\delta\mathbf{x}(t) = \exp(\lambda_i t)\mathbf{u}_i(t)$ into Eq. (24) and requiring that the adjoint vector be a T-periodic function,

$$\mathbf{z}(t) = \mathbf{z}(t+T), \tag{27}$$

we reveal that the bilinear form (24) for the nonzero Floquet modes becomes zero:

$$\begin{aligned}&\mathbf{z}^T(t)\mathbf{u}_i(t) + \int_{-\tau}^{0} \mathbf{z}^T(t+\tau+\vartheta)\mathbf{B}(t+\tau+\vartheta) \\ &\quad\times \mathbf{u}_i(t+\vartheta)e^{\lambda_i\vartheta}d\vartheta = 0, \quad i \geq 1.\end{aligned} \tag{28}$$

Since Eq. (23) is linear and admits solutions with arbitrary amplitudes, we can require that the bilinear form for the zero Floquet mode be equal to unity:

$$\begin{aligned}&\mathbf{z}^T(t)\mathbf{u}_0(t) + \int_{-\tau}^{0} \mathbf{z}^T(t+\tau+\vartheta)\mathbf{B}(t+\tau+\vartheta) \\ &\quad\times \mathbf{u}_0(t+\vartheta)d\vartheta = 1.\end{aligned} \tag{29}$$

We apply these useful equalities when considering the solution of the perturbed DDE system.

Now we define the phases for the states outside of the limit cycle. Although here we deal with an infinite-dimensional function space, the phase definition outside of the limit cycle introduced for ODE systems can be simply adjusted to the DDE system. We consider two states in the function space $\mathcal{C}$, one of them belonging to the limit cycle characterized by the phase φ, $\tilde{\boldsymbol{\chi}}^{(\varphi)}(\vartheta) = \tilde{\mathbf{x}}_\varphi(\vartheta) = \tilde{\mathbf{x}}(\varphi+\vartheta)$ and another outside of the limit cycle, which we denote as $\boldsymbol{\chi}^{(\varphi)}(\vartheta)$. We interpret the states $\tilde{\boldsymbol{\chi}}^{(\varphi)}(\vartheta)$ and $\boldsymbol{\chi}^{(\varphi)}(\vartheta)$ as two different initial conditions for the DDE (19) and consider their evolution in time. The evolution of the first state is defined by the limit cycle solution $\tilde{\mathbf{x}}_{\varphi+t}(\vartheta) = \tilde{\mathbf{x}}(\varphi+t+\vartheta)$. The evolution of the second state is defined by the solution of system (19) with the initial condition $\boldsymbol{\chi}^{(\varphi)}(\vartheta)$. We denote this solution as $\mathbf{x}_t(\vartheta)$. We say that the phase of the state $\boldsymbol{\chi}^{(\varphi)}(\vartheta)$ is φ (the same as that of the state $\tilde{\boldsymbol{\chi}}^{(\varphi)}(\vartheta)$ belonging to the limit cycle) if the evolutions of the two states asymptotically coincide, i.e. $\mathbf{x}_t(\vartheta) \to \tilde{\mathbf{x}}_{\varphi+t}(\vartheta) = \tilde{\mathbf{x}}(\varphi+t+\vartheta)$ for $t \to \infty$.

Using the Floquet theory, we can assign the phases for the states in the neighborhood of the limit cycle as follows. Let us take the state $\tilde{\boldsymbol{\chi}}^{(\varphi)}(\vartheta) = \tilde{\mathbf{x}}_\varphi(\vartheta)$ on the limit cycle with the phase φ and perturb it slightly in the "direction" of the ith Floquet mode

$$\boldsymbol{\chi}_i^{(\varphi)}(\vartheta) = \tilde{\mathbf{x}}_\varphi(\vartheta) + \varepsilon e^{\lambda_i\vartheta}\mathbf{u}_i(\varphi+\vartheta). \tag{30}$$

We suppose that ε is sufficiently small that the dynamics of the perturbation admits consideration by the linearized equation. Then the solution of Eq. (19) with the initial condition (30) takes the form

$$\mathbf{x}_t(\vartheta) = \tilde{\mathbf{x}}_{\varphi+t}(\vartheta) + \varepsilon e^{\lambda_i(t+\vartheta)}\mathbf{u}_i(\varphi+t+\vartheta). \tag{31}$$

If we consider the nonzero Floquet mode ($i \neq 0$), then $\Re(\lambda_i) < 0$ and the last term in the r.h.s. of Eq. (31) vanishes for $t \to \infty$. Thus according to the above definition, the state (30) has the phase φ. Since this conclusion holds for any $i \geq 1$, a general expression for the state with the phase φ can be presented in the form

$$\chi^{(\varphi)}(\vartheta) = \tilde{\mathbf{x}}_\varphi(\vartheta) + \varepsilon \sum_{i=1}^{\infty} c_i e^{\lambda_i \vartheta} \mathbf{u}_i(\varphi + \vartheta), \tag{32}$$

where c_i are arbitrary constants. If we fix φ and vary c_i, we obtain an isochron "surface" in the function space $\mathcal{C}$ close to the limit cycle state $\tilde{\mathbf{x}}_\varphi(\vartheta)$. Since the isochrons are dense, any state near the limit cycle can be represented by Eq. (32).

2.2.2. Phase reduction of the perturbed DDE system

Now we consider a weakly perturbed DDE system

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{x}(t-\tau)) + \varepsilon \boldsymbol{\psi}(t). \tag{33}$$

We suppose that under the action of the perturbation, the states of the system remain close to the limit cycle such that they can be parameterized by the phases of the unperturbed system according to Eq. (32). More specifically, we look for the solution of Eq. (33) in the form of Eq. (32) assuming that $\varphi = \varphi(t)$ and $c_i = c_i(t)$ are time-dependent functions:

$$\mathbf{x}(t+\vartheta) = \tilde{\mathbf{x}}(\varphi(t)+\vartheta) + \varepsilon \sum_{i=1}^{\infty} c_i(t) e^{\lambda_i \vartheta} \mathbf{u}_i(\varphi(t)+\vartheta),$$
$$\vartheta \in [-\tau, 0]. \tag{34}$$

Using Eqs. (33) and (34) we are going to derive a differential equation for the phase $\varphi(t)$. As has been stated above, the equation for the phase of the unperturbed ($\varepsilon = 0$) system is $d\varphi/dt = 1$. Our aim is to derive an equation for the phase with accuracy ε. Generally, this equation can be presented in the form

$$\frac{d\varphi}{dt} = 1 + \varepsilon q(\varphi, t), \tag{35}$$

where $q(\varphi, t)$ is an as yet unknown scalar function that has to be determined.

Taking $\vartheta = 0$ and $\vartheta = -\tau$ in Eq. (34) we get

$$\mathbf{x}(t) = \tilde{\mathbf{x}}(\varphi) + \varepsilon \sum_{i=1}^{\infty} c_i(t) \mathbf{u}_i(\varphi), \tag{36}$$

$$\mathbf{x}(t-\tau) = \tilde{\mathbf{x}}(\varphi-\tau) + \varepsilon \sum_{i=1}^{\infty} c_i(t) e^{-\lambda_i \tau} \mathbf{u}_i(\varphi-\tau). \tag{37}$$

Substituting these expressions into the r.h.s. of Eq. (33) and expanding the nonlinear function up to the first-order small terms we obtain

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\tilde{\mathbf{x}}(\varphi), \tilde{\mathbf{x}}(\varphi-\tau)) + \varepsilon \sum_{i=1}^{\infty} c_i(t)[\mathbf{A}(\varphi)\mathbf{u}_i(\varphi)$$
$$+ \mathbf{B}(\varphi-\tau)\mathbf{u}_i(\varphi-\tau)e^{-\lambda_i \tau}] + \varepsilon \boldsymbol{\psi}(t). \tag{38}$$

The first term in the r.h.s. of this equation represents the zero Floquet function: $\mathbf{F}(\tilde{\mathbf{x}}(\varphi), \tilde{\mathbf{x}}(\varphi-\tau)) = d\tilde{\mathbf{x}}(\varphi)/d\varphi = \mathbf{u}_0(\varphi)$. The expression in the square brackets can be simplified using Eq. (26). Then Eq. (38) transforms to

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}_0(\varphi) + \varepsilon \sum_{i=1}^{\infty} c_i(t) \left[\lambda_i \mathbf{u}_i(\varphi) + \frac{d\mathbf{u}_i}{d\varphi}\right] + \varepsilon \boldsymbol{\psi}(t). \tag{39}$$

Now we differentiate Eq. (36) with respect to φ:

$$\frac{d\mathbf{x}}{dt}\frac{dt}{d\varphi} = \mathbf{u}_0(\varphi) + \varepsilon \sum_{i=1}^{\infty} \left[\frac{dc_i}{dt}\frac{dt}{d\varphi}\mathbf{u}_i(\varphi) + c_i(t)\frac{d\mathbf{u}_i}{d\varphi}\right]. \tag{40}$$

Excluding $d\mathbf{x}/dt$ from Eqs. (39) and (40) we get

$$\mathbf{u}_0(\varphi) + \varepsilon \sum_{i=1}^{\infty} c_i(t) \left[\lambda_i \mathbf{u}_i(\varphi) + \frac{d\mathbf{u}_i}{d\varphi}\right] + \varepsilon \boldsymbol{\psi}(t)$$
$$= \frac{d\varphi}{dt}\left[\mathbf{u}_0(\varphi) + \varepsilon \sum_{i=1}^{\infty} c_i(t)\frac{d\mathbf{u}_i}{d\varphi}\right] + \varepsilon \sum_{i=1}^{\infty} \frac{dc_i}{dt}\mathbf{u}_i(\varphi). \tag{41}$$

Substituting $d\varphi/dt = 1 + \varepsilon q(\varphi, t)$ from Eq. (35) and omitting the terms of order higher than ε we simplify this equation as follows:

$$q(\varphi, t)\mathbf{u}_0(\varphi) = \sum_{i=1}^{\infty} \mathbf{u}_i(\varphi)\left[\lambda_i c_i(t) - \frac{dc_i}{dt}\right] + \boldsymbol{\psi}(t). \tag{42}$$

As a next step we consider the derivatives of Eq. (34) with respect to ϑ and t. It is easy to see that the derivatives of the l.h.s. of Eq. (34) with respect to ϑ and t are equal to each other. Thus, the same conclusion is valid for the r.h.s. of Eq. (34). Differentiating the r.h.s. of Eq. (34) with respect to ϑ and t and equating these derivatives, we get

$$\mathbf{u}_0(\varphi+\vartheta) + \varepsilon \sum_{i=1}^{\infty} c_i(t) e^{\lambda_i \vartheta}\left[\lambda_i \mathbf{u}_i(\varphi+\vartheta) + \frac{d\mathbf{u}_i(\varphi+\vartheta)}{d\varphi}\right]$$
$$= \frac{d\varphi}{dt}\left[\mathbf{u}_0(\varphi+\vartheta) + \varepsilon \sum_{i=1}^{\infty} c_i(t)\frac{d\mathbf{u}_i(\varphi+\vartheta)}{d\varphi}e^{\lambda_i \vartheta}\right]$$
$$+ \varepsilon \sum_{i=1}^{\infty} \frac{dc_i}{dt}\mathbf{u}_i(\varphi+\vartheta)e^{\lambda_i \vartheta}, \quad \vartheta \in (-\tau, 0). \tag{43}$$

This equation is valid in the open interval of the variable $\vartheta \in (-\tau, 0)$ since the derivatives of Eq. (34) are not defined on the borders of the interval $\vartheta = -\tau$ and $\vartheta = 0$. To simplify Eq. (43) we again use the substitution (35) and omit the terms higher than ε:

$$q(\varphi, t)\mathbf{u}_0(\varphi+\vartheta) = \sum_{i=1}^{\infty} e^{\lambda_i \vartheta}\mathbf{u}_i(\varphi+\vartheta)\left[\lambda_i c_i(t) - \frac{dc_i}{dt}\right],$$
$$\vartheta \in (-\tau, 0). \tag{44}$$

Now using the properties of the bilinear form (28) and (29), from Eqs. (42) and (44) we can derive a simple expression for the unknown function $q(\varphi, t)$. To this end we multiply Eq. (44) from the left-hand side by $\mathbf{z}^T(\varphi+\tau+\vartheta)\mathbf{B}(\varphi+\tau+\vartheta)$ and integrate it with respect to ϑ in the interval $(-\tau, 0)$. The result of the integration we add to Eq. (42) multiplied from the left-hand side by $\mathbf{z}^T(\varphi)$. Then using the properties (28) and (29) we get

$$q(\varphi, t) = \mathbf{z}^T(\varphi)\boldsymbol{\psi}(t). \tag{45}$$

Substituting this expression into Eq. (35) we obtain finally the equation for the phase

$$\frac{d\varphi}{dt} = 1 + \varepsilon \mathbf{z}^T(\varphi)\boldsymbol{\psi}(t), \tag{46}$$

where $\mathbf{z}(\varphi)$ is a T-periodic solution of the adjoint equation (23). The initial condition for the adjoint equation can be derived from Eq. (29) by substituting $t = 0$ and taking it into account that $\mathbf{u}_0(t) = \dot{\tilde{\mathbf{x}}}(t)$:

$$\mathbf{z}^T(0)\dot{\tilde{\mathbf{x}}}(0) + \int_{-\tau}^{0} \mathbf{z}^T(\tau+\vartheta)\mathbf{B}(\tau+\vartheta)\dot{\tilde{\mathbf{x}}}(\vartheta)d\vartheta = 1. \tag{47}$$

Thus, the phase dynamics of the DDE system is completely determined by (23), (46) and (47). These are equivalent respectively to Eqs. (17), (14) and (16) derived in Section 2.1.

3. Numerical demonstrations for the Mackey–Glass equation

Now we support the validity of the above theoretical results and demonstrate their efficiency using a simple example of a time-delay system described by the Mackey–Glass (MG) equation [2]:

$$\frac{dx}{dt} = \frac{ax(t-\tau)}{1+x^b(t-\tau)} - x(t) + \varepsilon\psi(t). \tag{48}$$

This equation was initially introduced as a model of blood generation for patients with leukemia. Later, this equation became popular in chaos theory as a model for producing high-dimensional chaos to test various methods of chaotic time-series analysis, controlling chaos, etc. An electronic analog of this system has been proposed in Ref. [37]. Depending on the parameters, the MG equation can exhibit a reach variety of dynamical regimes. If we fix the parameters $a = 2, b = 10$ and increase the delay time τ we first observe a period doubling scenario, then chaos and, afterwards, a hyperchaos with a continually increasing dimension of the strange attractor. Here, we choose the parameters $a = 2$, $b = 10$ and $\tau = 0.7$ such that for $\varepsilon = 0$ the MG equation demonstrates a stable limit cycle behavior with the period $T \approx 2.29584$.

3.1. Comparison of two methods for the PRC computation

First, we verify whether the periodic solution of the adjoint equation (14) leads to the correct PRC. To this end, we compare this solution with the PRC definition based on the computation of the oscillator response to small short pulses delivered at different phases of oscillations.

To compute the PRC according to its definition, we proceed as follows. First for the given values of the parameters we integrate a free ($\varepsilon = 0$) MG equation (48) for sufficiently long time until the periodic solution $\tilde{x}(t) = \tilde{x}(t+T)$ corresponding to the stable limit cycle is established. This solution $\tilde{x}(\varphi)$ is depicted in Fig. 1(a) as a function of the phase φ, taking into account that $\varphi = t$ on the limit cycle. Then for a given phase φ we perturb the system by a small short pulse $\varepsilon\psi(t) = \varepsilon\delta(t+\varphi)$, where $\delta(t)$ is the Dirac delta function. As pointed out in Section 2.2, the state of the DDE system on the limit cycle is determined by the function $\tilde{x}(\varphi+\vartheta)$, $\vartheta \in [-\tau, 0]$. Thus, the perturbed state for a given phase is

$$\chi(\varphi+\vartheta) = \begin{cases} \tilde{x}(\varphi+\vartheta)+\varepsilon & \text{for } \vartheta = 0, \\ \tilde{x}(\varphi+\vartheta) & \text{for } \vartheta \in [-\tau, 0). \end{cases} \tag{49}$$

We take the perturbed state (49) as a new initial condition for the MG equation (48) and integrate it for several periods mT (here m is a sufficiently large integer number) until the solution approaches the limit cycle,

$$x(\varphi+\vartheta+mT) \to \tilde{x}(\varphi+\Delta\varphi+\vartheta), \quad \vartheta \in [-\tau, 0]. \tag{50}$$

Then we estimate the phase shift $\Delta\varphi$ of the perturbed solution in comparison with the unperturbed solution. The value of the infinitesimal PRC at the phase φ is by definition equal to the ratio $\Delta\varphi/\varepsilon$ for $\varepsilon \to 0$. The results of such a computation for $\varepsilon = 10^{-5}$ are represented in Fig. 1(b) by circles.

In Fig. 1(b), we compare this result with the PRC computed from the adjoint equation (14). Note that the Jacobians (15) for the MG equation are the scalar functions

$$\mathbf{A}(t) = -1, \qquad \mathbf{B}(t) = a\frac{1+(1-b)\tilde{x}^b(t-\tau)}{[1+\tilde{x}^b(t-\tau)]^2}. \tag{51}$$

The periodic solution of the adjoint equation (14) satisfying the initial condition (16) has been obtained via its backward integration in a way described in Section 2.1.3. This solution is represented in Fig. 1(b) by a solid curve. As is seen from the figure,

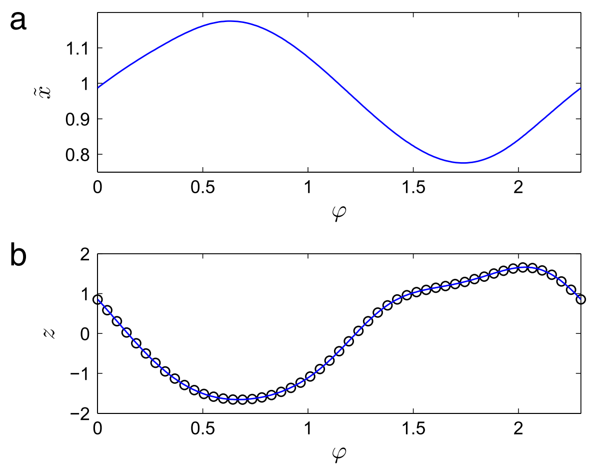

Fig. 1. (Color online) (a) The limit cycle solution and (b) the PRC of the MG equation (48) for $a = 2, b = 10$ and $\tau = 0.7$. Circles in (b) show the values of the PRC derived from the system response to small δ-pulses according to Eqs. (49) and (50), while the solid curve represents the solution of the adjoint equation (14).

both methods of PRC computation lead to the same results. This supports the validity of the adjoint equation (14) and the initial condition (16) derived in the previous section.

Note that the second method of PRC computation, based on the solution of the adjoint equation (14), has a significant advantage over the first method, since generally it provides a more accurate estimate of the infinitesimal PRC. This is because the adjoint equation describes the linearized system's dynamics, and thus it deals with the infinitesimal perturbations, while the first method uses the finite perturbations.

3.2. Arnold tongues of the periodically perturbed MG equation

To verify the validity of the reduced phase Eq. (17) we consider the problem of synchronization of the MG equation with the harmonic $\varepsilon\psi(t) = \varepsilon\sin(2\pi\nu t)$ as well as the rectangular $\varepsilon\psi(t) = \varepsilon\,\mathrm{sign}[\sin(2\pi\nu t)]$ external periodic perturbation. The frequency ν of the perturbation is assumed to be close to the frequency $1/T$ of the limit cycle. We analyze the dependence of the threshold amplitude $\varepsilon = \varepsilon_c$ of the synchronization on the frequency mismatch $\Delta\nu = \nu - 1/T$ (the Arnold tongues) by two different methods, namely, by the direct integration of the original time-delay equation (48) and by the solution of the reduced phase Eq. (17) with the PRC $z(\varphi)$ estimated from the adjoint equation (14). If the frequency mismatch is small, the phase equation can be averaged over the period of the external force. Then the second method leads to the following simple expression for the Arnold tongue:

$$\varepsilon_c = \begin{cases} \Delta\nu T/\min[I(\varphi)] & \text{for } \Delta\nu \le 0, \\ \Delta\nu T/\max[I(\varphi)] & \text{for } \Delta\nu > 0. \end{cases} \tag{52}$$

Here $\min[I(\varphi)]$ and $\max[I(\varphi)]$ denote respectively the minimal and maximal values of the periodic function $I(\varphi) = I(\varphi+T)$ defined as

$$I(\varphi) = \frac{1}{\Theta}\int_0^{\Theta} z\left(\varphi + \frac{T}{\Theta}t\right)\psi(t)dt, \tag{53}$$

where $\Theta = 1/\nu$ is the period of the external perturbation, $\psi(t) = \psi(t+\Theta)$.

In Fig. 2, the Arnold tongues determined by the first and second methods are represented by circles and straight lines, respectively. As is seen from the figure, the two methods produce the same results if the frequency mismatch is small. This supports the

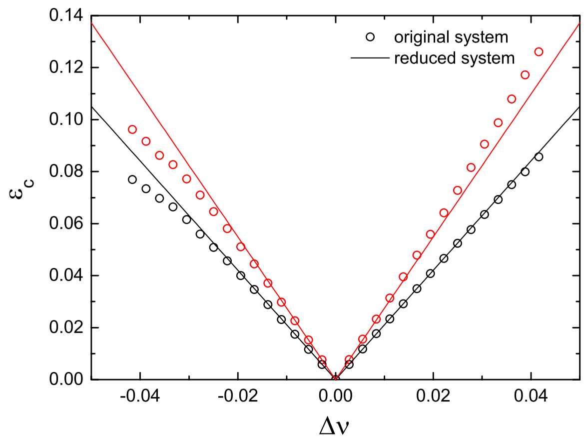

Fig. 2. (Color on-line) Arnold tongues (the dependence of the threshold synchronization amplitude ε_c on the frequency mismatch $\Delta\nu = 1/T - \nu$) of the Mackey–Glass equation perturbed by the sinusoidal (red) and rectangular (black) waves. Circles show the results of numerical simulation of the original time-delay equation (48). Lines are depicted according to Eq. (52), which is derived from the reduced phase Eq. (17) with the PRC estimated from Eqs. (14)–(16).

validity of the whole phase reduction procedure performed in Sections 2.1 and 2.2.

The advantage of the phase Eq. (17) over the original time-delay equation (48) is that it allows a simple analysis of the system response to any weak time-dependent perturbation.

4. Applications to chaotic systems controlled by the DFC algorithm

Unstable periodic orbits (UPOs) embedded in strange attractors of chaotic systems can be stabilized by the delayed feedback control (DFC) method [9]. The control signal in the DFC algorithm is formed from a difference between the current state of the system, and the state of the system delayed by one period of a target orbit. Such a control signal allows one to treat the controlled system as a black box; it does not require any exact knowledge of either the profile of the periodic orbit or the system's equations. The method is asymptotically noninvasive because the control force vanishes whenever the target UPO is reached. The DFC algorithm has been successfully implemented in quite diverse experimental systems from different fields of science. Some details of experimental implementations as well as various modifications of the DFC algorithm can be found in the review paper [10].

4.1. General properties of the PRC for a system subject to the DFC

Here, we consider the influence of small perturbations on the dynamics of chaotic systems controlled by the DFC algorithm. Specifically, suppose that we have a chaotic system described by the ODE $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t))$, where $\mathbf{x}(t)$ is an n-dimensional state vector. Assume that a strange attractor of this system has an UPO $\tilde{\mathbf{x}}(t) = \tilde{\mathbf{x}}(t+T)$ with a period T. To stabilize this UPO, we apply a DFC force in the form $\mathbf{K}[\mathbf{x}(t-\tau) - \mathbf{x}(t)]$, where $\mathbf{K}$ is a control matrix and the delay time τ is taken equal to the period of the UPO, $\tau = T$. Then, after the addition of a small time-dependent perturbation $\varepsilon\boldsymbol{\psi}(t)$ our system can be represented by the following DDE:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{K}[\mathbf{x}(t-\tau) - \mathbf{x}(t)] + \varepsilon\boldsymbol{\psi}(t). \quad (54)$$

If the control matrix $\mathbf{K}$ is chosen appropriately, the control force $\mathbf{K}[\mathbf{x}(t-\tau) - \mathbf{x}(t)]$ can stabilize the previously unstable periodic orbit. In the following, we suppose that $\mathbf{K}$ is chosen such that the given UPO $\tilde{\mathbf{x}}(t)$ represents the stable limit cycle solution of the DDE (54) for $\varepsilon = 0$. Then we can apply the above theory in order to analyze an influence of the small perturbation $\varepsilon\boldsymbol{\psi}(t)$ on the phase dynamics of the stabilized orbit.

First, we analyze the PRC of the system (54) and show its interesting properties. The PRC is defined by the adjoint equation (14). The matrix equations (15) for our system are $\mathbf{A}(t) = D\mathbf{f}(\tilde{\mathbf{x}}(t)) - \mathbf{K}$, $\mathbf{B}(t) = \mathbf{K}$ and the adjoint equation (14) takes the form

$$\dot{\mathbf{z}}^T(t) = -\mathbf{z}^T(t)\mathbf{A}_0(t) - [\mathbf{z}^T(t+\tau) - \mathbf{z}^T(t)]\mathbf{K}, \quad (55)$$

where $\mathbf{A}_0(t) = D\mathbf{f}(\tilde{\mathbf{x}}(t))$ is the Jacobian of the control-free system estimated on the UPO $\tilde{\mathbf{x}}(t)$. The initial condition (16) for this system becomes

$$\mathbf{z}^T(0)\dot{\tilde{\mathbf{x}}}(0) + \int_{-\tau}^{0} \mathbf{z}^T(\tau+\vartheta)\mathbf{K}\dot{\tilde{\mathbf{x}}}(\vartheta)d\vartheta = 1. \quad (56)$$

The PRC of the stabilized UPO is the periodic solution $\mathbf{z}^T(t) = \mathbf{z}^T(t+T)$ of Eq. (55). Since $\tau = T$ the last term in Eq. (55) vanishes for the periodic solution and thus the PRC of the controlled UPO also satisfies the adjoint equation of the control-free system:

$$\dot{\mathbf{z}}^T(t) = -\mathbf{z}^T(t)\mathbf{A}_0(t). \quad (57)$$

Unfortunately, this equation is difficult to employ for a numerical computation of the PRC, since it is unstable for both the backward and the forward integration. This is because any UPO of a chaotic system contains the Lyapunov exponents with the positive and negative real parts. To obtain the PRC for a specific value of $\mathbf{K}$ we can use Eq. (55), since it is stable for the backward integration provided $\mathbf{K}$ is chosen from the domain of stability of the given UPO. Although Eq. (57) is inappropriate for numerical integration, it allows us to reach an important conclusion regarding the profile of the PRC. Since Eq. (57) is independent of the control matrix $\mathbf{K}$, the profile of the PRC is invariant with respect to the variation of $\mathbf{K}$. The variation of $\mathbf{K}$ changes only the amplitude of the PRC. These changes can be estimated from the initial condition (56) and we do not need to integrate Eq. (55) for each different value of $\mathbf{K}$. Specifically, suppose that using Eqs. (55) and (56) we have obtained the PRC $\mathbf{z}^{(1)}(\varphi)$ of the controlled orbit for some appropriate control matrix $\mathbf{K}_1$. Then the PRC $\mathbf{z}^{(2)}(\varphi)$ for another control matrix $\mathbf{K}_2$ is proportional to $\mathbf{z}^{(1)}(\varphi)$:

$$\mathbf{z}^{(2)}(\varphi) = \alpha_{1,2}\mathbf{z}^{(1)}(\varphi), \quad (58)$$

where the coefficient of proportionality $\alpha_{1,2}$ can be determined from Eq. (56):

$$\alpha_{1,2} = \left[\mathbf{z}^{(1)^T}(0)\dot{\tilde{\mathbf{x}}}(0) + \int_{-\tau}^{0} \mathbf{z}^{(1)^T}(\tau+\vartheta)\mathbf{K}_2\dot{\tilde{\mathbf{x}}}(\vartheta)d\vartheta\right]^{-1}. \quad (59)$$

4.2. A demonstration for the Rössler system

As an example of application of the above theory we consider the Rössler system [38] subject to the DFC and a weak external perturbation:

$$\dot{x}_1(t) = -x_2(t) - x_3(t), \quad (60a)$$

$$\dot{x}_2(t) = x_1(t) + ax_2(t) + K[x_2(t-\tau) - x_2(t)], \quad (60b)$$

$$\dot{x}_3(t) = b + x_3(t)[x_1(t) - c] + \varepsilon. \quad (60c)$$

Here, we suppose that a measurable control signal is $x_2(t)$ and the DFC force is applied only to the second equation of the Rössler system, i.e. the control term is defined by a diagonal matrix of the form $\mathbf{K} = \mathrm{diag}(0, K, 0)$. We consider a simple case of a constant external perturbation and suppose that it is applied only to the third equation, i.e. we take $\boldsymbol{\psi}(t) = (0, 0, 1)^T$. We choose the parameters of the Rössler system $a = 0.2$, $b = 0.2$ and $c = 5.7$ such that for $K = 0$ and $\varepsilon = 0$ it exhibits a chaotic behavior.

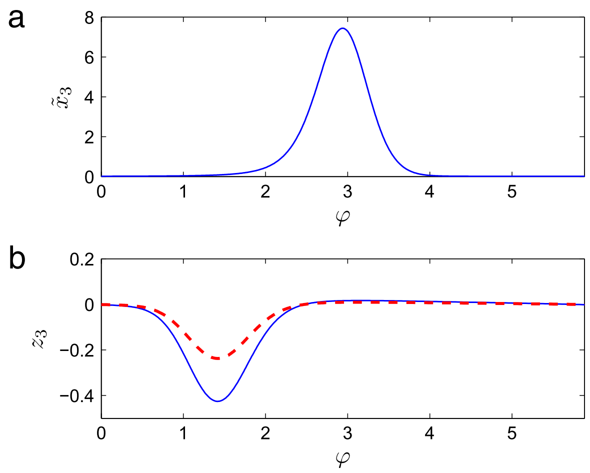

Fig. 3. (Color on-line) (a) The third component of the period-1 UPO and (b) the third component of the PRC of the Rössler system (60) subject to the DFC for $a = 0.2$, $b = 0.2$ and $c = 5.7$. The blue solid and red dashed curves in (b) correspond to the control gains $K_1 = 0.15$ and $K_2 = 0.5$, respectively.

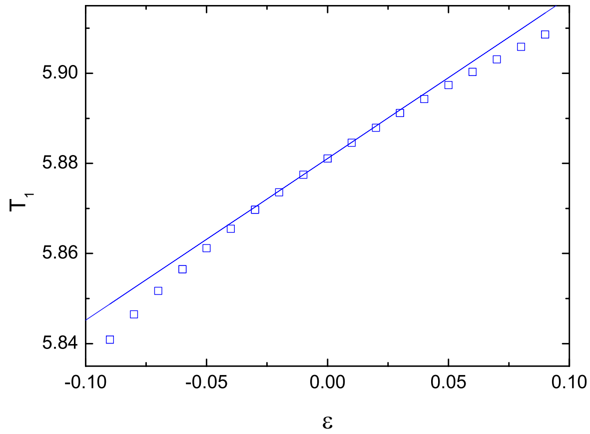

Fig. 4. The dependence of the period T_1 of the stabilized period-1 UPO of the Rössler system (60) on the constant perturbation ε. The solid curve corresponds to Eq. (62) derived from the reduced phase Eq. (17), while the squares show the results obtained from the direct integration of the original system (60).

An approximate period of the period-1 UPO, which we intend to stabilize, is $T \approx 5.881$. The third component $\tilde{x}_3$ of this UPO is depicted in Fig. 3(a).

First, we take $\varepsilon = 0$ and by backward integration of the adjoint equation (55) with the initial condition (56) we compute two PRCs $\mathbf{z}^{(1)}(\varphi)$ and $\mathbf{z}^{(2)}(\varphi)$ of the stabilized UPO for two different values of the coupling strength, $K_1 = 0.15$ and $K_2 = 0.5$, respectively. In Fig. 3, we present the third components of these PRCs. As is expected from the general theory, these PRCs are proportional to each other. An approximate value of the proportionality coefficient is $\alpha_{1,2} \approx 0.558$. On the other hand, the same value of the coefficient is obtained from Eqs. (56) and (59):

$$\alpha_{1,2} = \left[{\mathbf{z}^{(1)}}^T(0)\dot{\tilde{\mathbf{x}}}(0) + \frac{K_2}{K_1}\left(1 - {\mathbf{z}^{(1)}}^T(0)\dot{\tilde{\mathbf{x}}}(0)\right) \right]^{-1}. \tag{61}$$

This supports the validity of the general relationship (58) and Eq. (59).

As a next step we consider an influence of a small constant perturbation $\varepsilon \neq 0$ to the stabilized UPO. Our aim is to change the period of the stabilized UPO by the variation of the parameter ε. Knowing the PRC, the value of the perturbed period $T_1(\varepsilon)$ can be easily estimated by the integration of the phase Eq. (17):

$$T_1 = \int_0^T \frac{d\varphi}{1 + \varepsilon z_3(\varphi)} \approx T - \varepsilon \int_0^T z_3(\varphi) d\varphi. \tag{62}$$

Certainly, this estimate is valid only for small values of the parameter ε. The precise value of the perturbed period $T_1(\varepsilon)$ can be obtained by direct integration of the original system (60). The results from the two methods are compared in Fig. 4. We see that for small values of ε, the approximation (62) based on the PRC theory provides the correct result.

5. Conclusions

In this paper, we have systematically developed a phase reduction procedure for a general class of weakly perturbed time-delay systems exhibiting periodic oscillations. In our approach, we assume the delay terms to be large, such that they are essential for the formation of the limit cycle and consequently of the phase response curve (PRC). This differs from previous consideration of phase reduced models with time delay [28–33], where the delay terms have been considered as small perturbations to a delay-free system.

We have presented two methods of derivation of the phase reduced equation for the time-delay systems. One of them is based on the approximation of a time-delay system using a finite set of ordinary differential equations (ODEs) and using the known results from the ODE theory. The second method initially deals with the infinite-dimensional phase space, and the phase reduced equations are directly derived from the time-delay system. We have shown that the infinitesimal PRC of a time-delay system satisfies the adjoint equation, which represents a difference–differential equation of the advanced type. Although this equation is unstable, its periodic solution can be obtained numerically by a backward integration.

The general theoretical results are supported by specific examples. We have demonstrated the efficiency of a phase reduction procedure for estimating the Arnold tongues for a periodically perturbed Mackey–Glass equation. The phase reduction method has also been applied to chaotic systems subject to the delayed feedback control (DFC) force and a weak external time-dependent perturbation. We have revealed an interesting general property that the profile of the PRC of a periodic orbit stabilized by the DFC is invariant under any variation of the control matrix. This property has been supported numerically for the Rössler system. The phase reduction theory also suggests a simple algorithm for evaluating the effect of a weak perturbation on the period of a stabilized orbit.

Although, in this paper, we have restricted ourselves to the systems with a single time delay, an extension of our approach to systems with multiple time delays is straightforward.

Acknowledgments

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