Suitable interaction picture for the high-frequency expansion of periodic Hamiltonians belonging to $\mathfrak{su}(3)$ Lie algebra

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Main background

Quantum system described by a Hamiltonian $h(\omega t+\theta,t)=\sum_{n=-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}n(\omega t+\theta)}h^{(n)}(t)$ which is periodic with respect to the first argument and has additional slow time dependence:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\left|\psi_\theta(t)\right\rangle=h(\omega t+\theta,t)\left|\psi_\theta(t)\right\rangle \qquad (*)$$

Expanding $\left|\psi_\theta(t)\right\rangle=\sum_n \mathrm{e}^{\mathrm{i}n\theta}\left|\psi^{(n)}(t)\right\rangle$ and using extended space approach [1] $\mathscr{L}=\mathscr{T}\otimes\mathscr{H}$, where $\mathrm{e}^{\mathrm{i}n\theta}\equiv|n\rangle\in\mathscr{T}$ is orthonormal basis, we transform Eq (*) into:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\phi(t)\rangle\rangle=\mathcal{K}(t)|\phi(t)\rangle\rangle$$

with $|\phi(t)\rangle\rangle=\mathcal{U}^\dagger|\psi(t)\rangle\rangle=\mathrm{e}^{-\omega t\frac{\mathrm{d}}{\mathrm{d}\theta}}|\psi(t)\rangle\rangle$ and

$$\mathcal{K}(t)=\sum_n|n\rangle n\hbar\omega\langle n|\otimes\mathbf{1}_{\mathscr{H}}+\sum_{n,m}|m\rangle\langle n|\otimes h^{(m-n)}(t)$$

The main task is to find block-diagonalizing operator $\mathcal{D}(t)$ such that

$$\mathcal{K}_D(t)=\mathcal{D}^\dagger\mathcal{K}\mathcal{D}-\mathrm{i}\hbar\mathcal{D}^\dagger\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}t}$$

contains non-zero blocks only on a central diagonal [2]:

$$\mathcal{K}_D(t)=\sum_n|n\rangle n\hbar\omega\langle n|\otimes\mathbf{1}_{\mathscr{H}}+\sum_n|n\rangle\langle n|\otimes h_{\mathrm{eff}}(t)$$

By expanding the operator $\mathcal{D}(t)$ as a power series in terms of the inverse frequency, we obtain effective Hamiltonian:

$$h_{\mathrm{eff}(0)}(t)=h^{(0)}(t)\,,$$
$$h_{\mathrm{eff}(1)}(t)=\frac{1}{\hbar\omega}\sum_{m=1}^{+\infty}\frac{\left[h^{(m)}(t),h^{(-m)}(t)\right]}{m}. \qquad (\#)$$

Problem formulation

If $h^{(m)}(t)\sim\mathcal{O}\left((\hbar\omega)^1\right)$ the expansion (#) diverges. Before applying the block-diagonalization $\mathcal{D}(t)$ one can reduce the order of our Hamiltonian to $\mathcal{O}(1)$ by using interaction picture defined by some unitary transformation

$$\mathcal{R}(t)=\exp\left[-\mathrm{i}\sum_{m,n}|m\rangle\langle n|\otimes a^{(m-n)}(t)\right].$$

The Hamiltonian in the interaction picture reads

$$\mathcal{K}_R(t)=\sum_n|n\rangle n\hbar\omega\langle n|\otimes\mathbf{1}_{\mathscr{H}}-\mathrm{i}\hbar\mathcal{R}^\dagger(t)\frac{\mathrm{d}\mathcal{R}(t)}{\mathrm{d}t}$$
$$+\hbar\omega\sum_{k=0}^{+\infty}\frac{\mathrm{i}^k}{k!}\mathrm{ad}^k_{\sum_{m,n}|m\rangle\langle n|\otimes a^{(m-n)}(t)}\sum_{p,q}|p\rangle\langle q|$$
$$\otimes\left\{h^{(p-q)}(t)-\frac{\mathrm{i}(p-q)}{k+1}a^{(p-q)}(t)\right\}$$

term represents non-Abelian geometric phase

How to choose $a^{(m)}(t)$ such that the **blue** term vanishes?

For the special case

$$h^{(m)}(t)=\hat{H}(t)\,g^{(m)}$$

operator number

$$a^{(m\neq0)}(t)=\hat{H}(t)\frac{g^{(m)}}{\mathrm{i}m}$$
$$a^{(0)}(t)=0$$

Non-Abelian geometric phase

The Hamiltonian describing the non-Abelian geometric dynamics obtained as a zero order effective Hamiltonian:

$$\langle 0|-\mathrm{i}\hbar\mathcal{R}^\dagger(t)\frac{\mathrm{d}\mathcal{R}(t)}{\mathrm{d}t}|0\rangle=-\hbar\sum_{k=0}^{+\infty}\frac{\mathrm{i}^k\left\langle G^{k+1}(\theta)\right\rangle}{(k+1)!}\mathrm{ad}^k_{\hat{H}(t)}\frac{\mathrm{d}\hat{H}(t)}{\mathrm{d}t}$$

For the one harmonic case, $G(\theta)=\sin\theta$, and $\mathfrak{su}(2)$ Lie algebra the effective Hamiltonian reads [3]:

$$h_{\mathrm{eff}(0)}(t)=\frac{1-J_0(g_F B/\omega)}{B^2}\left[\mathbf{F}\times\mathbf{B}(t)\right]\cdot\frac{\mathrm{d}\mathbf{B}(t)}{\mathrm{d}t}$$

See proposed experiment [4] and experimental realization [5].

The case of $\mathfrak{su}(3)$ Lie algebra

For the 8 dimensional Lie algebra $\mathfrak{su}(3)$ we need to perform a root space decomposition, assuming that at each time moment the vector $\hat{H}(t)\in\mathfrak{h}(t)$ is in the 2 dimensional Cartan subalgebra:

$$\mathfrak{su}(3)=\mathfrak{h}(t)\oplus\bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha(t)\,,$$

where $\Delta=\{\pm\alpha_1,\pm\alpha_2,\pm\alpha_3\}$ is 6 roots. For each root space $\hat{V}_{\pm\alpha}(t)\in\mathfrak{g}_{\pm\alpha}(t)$, the commutator

$$\left[\hat{H}(t),\hat{V}_{\pm\alpha}(t)\right]=\pm\alpha\hat{V}_{\pm\alpha}(t)\,.$$

Moreover, $\hat{V}_{\pm\alpha_j}(t)=\hat{X}_j(t)\pm\mathrm{i}\hat{Y}_j(t)$.

Therefore, the series of nested commutators gives

$$h_{\mathrm{eff}(0)}(t)=2\hbar\sum_{j=1}^{3}\frac{1-J_0(\alpha_j(t))}{\alpha_j(t)}\left[x_j\hat{X}_j(t)+y_j\hat{Y}_j(t)\right],$$

where $x_j=\hat{X}_j(t)\cdot\dfrac{\mathrm{d}\hat{H}(t)}{\mathrm{d}t}$ and $y_j=\hat{Y}_j(t)\cdot\dfrac{\mathrm{d}\hat{H}(t)}{\mathrm{d}t}$.

References

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