CENTER
FOR PHYSICAL SCIENCES
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Phase response curves for systems with time delay

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- Phase reduction of time-delay systems
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- Phase reduction of chaotic systems subjected to a DFC
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Introduction

Phase reduction method is an efficient tool to analyze weakly perturbed limit cycle oscillations

Most investigations in the field of phase reduction are devoted to the systems described by ODEs

The aim of this investigation is to extend the phase reduction method to time delay systems

Phase reduction

A dynamical system with a stable limit cycle

For each state on the limit cycle and near the limit cycle is assigned a scalar variable (PHASE)

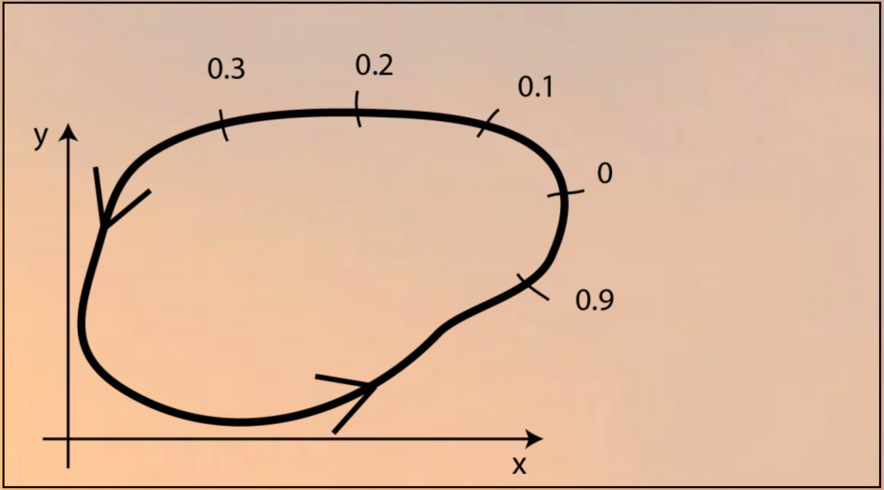
The phase dynamics of the free system satisfies:

$$\boxed{\dot{\varphi} = 1}$$

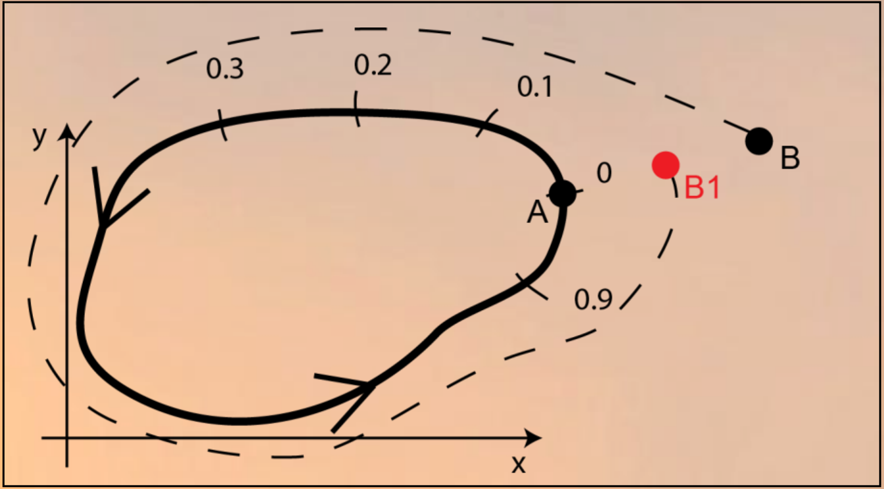
Let's apply an external perturbation to the system.
The aim of phase reduction method is to find a dynamical equation the phase of perturbed system:

$$\boxed{\dot{\varphi} = ?}$$

Phase reduction of ODE systems



Phase reduction of ODE systems



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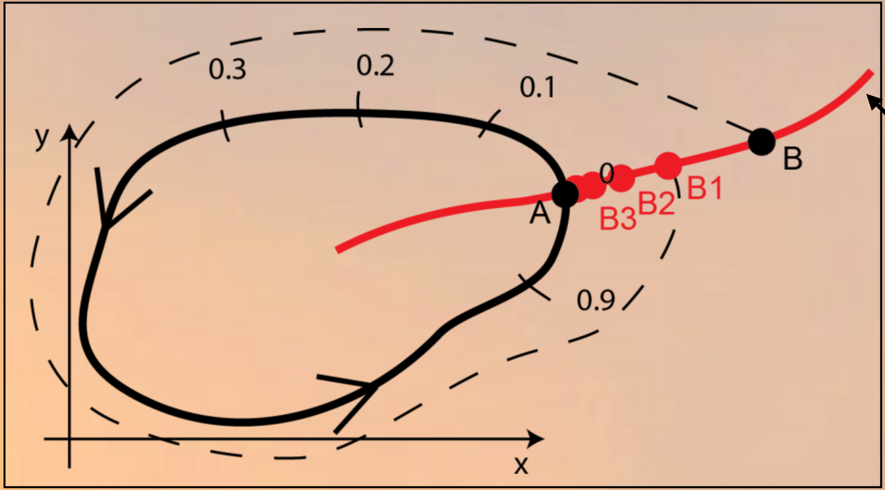
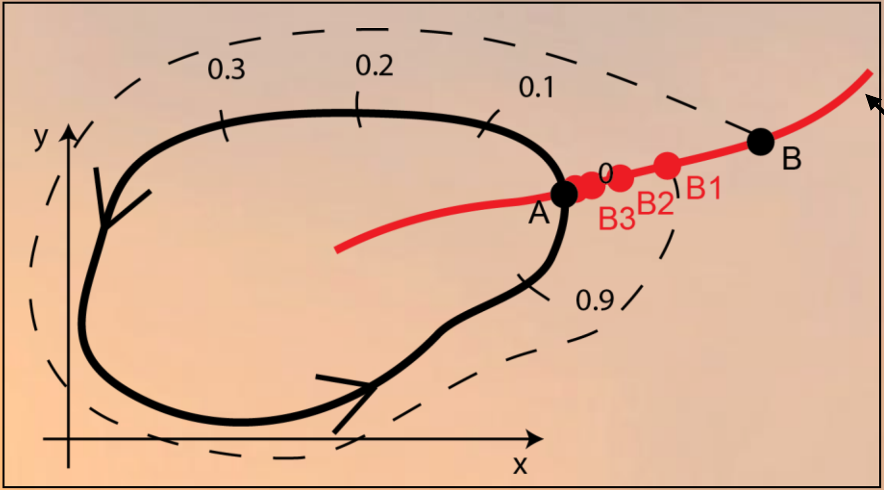
Phase reduction of ODE systems



$$\varphi_B = \varphi_A$$

Isochron – a set of states with the same phase

Phase reduction of ODE systems



$$\varphi_B = \varphi_A$$

Isochron – a set of states with the same phase

Malkin's approach:

Malkin, I.G.: Some Problems in Nonlinear Oscillation Theory.Gostexizdat, Moscow (1956)

Perturbed system: $\boxed{\dot{y} = G(y) + \varepsilon\phi(t)}$

Phase dynamics: $\boxed{\dot{\varphi} = 1 + \varepsilon z(\varphi)^T \phi(t)}$,here $z(\varphi)$ is periodic vector valued function - the phase response curve (PRC)

PRC is the periodic solution of an adjoint equation: $\boxed{\dot{z} = -[DG(y_c)]^T z}$

With initial condition: $\boxed{z^T(0)\dot{y}_c(0) = 1}$

Phase reduction of time-delay system

Perturbed system: $\dot{x} = F\left(x(t), x(t-\tau)\right) + \varepsilon\psi(t)$

Approximation via a delay line:

$$\dot{x} = F\left(x(t), \xi(\tau, t)\right) + \varepsilon\psi(t)$$

$$\frac{\partial\xi(s,t)}{\partial t} = -\frac{\partial\xi(s,t)}{\partial s}, \xi(0,t) = x(t)$$

Discretization of the space variable : $s_i = i\tau / N, \quad i = 0,..,N$

Denote $x_0(t) = x(t)$ and $x_i(t) = \xi(s_i, t)$

We get a final-dimensional system of ODEs:

$$\begin{aligned}
\dot{x}_0 &= F\left(x_0(t), x_N(t)\right) + \varepsilon\psi(t) \\
\dot{x}_1 &= \left[x_0(t) - x_1(t)\right] N / \tau \\
&\dots \\
\dot{x}_N &= \left[x_{N-1}(t) - x_N(t)\right] N / \tau
\end{aligned}$$

Phase reduction of time-delay system: results

Phase dynamic: $\dot{\varphi} = 1 + \varepsilon z^T(\varphi)\psi(t)$

The adjoint equation for PRC: $\dot{z} = -A^T(t)z(t) - B^T(t+\tau)z(t+\tau)$

here
$$A(t) = D_1 F\big(x_c(t), x_c(t-\tau)\big)$$
$$B(t) = D_2 F\big(x_c(t), x_c(t-\tau)\big)$$

An unstable difference-differential equation of advanced type (backwards integration)

The initial condition:

$$z^T(0)\dot{x}_c(0) + \int_{-\tau}^{0} z^T(\tau+\vartheta)B(\tau+\vartheta)\dot{x}_c(\tau+\vartheta)d\vartheta = 1$$

The phase reduced equations for time delay systems have been alternatively derive directly from DDE system without appealing to the known theoretical results from ODEs

Example: Mackey-Glass equation

Unperturbed equation: $$\frac{dx}{dt} = \frac{ax(t-\tau)}{1+x^{b}(t-\tau)} - x(t)$$

Two different initial conditions: first on the limit cycle and second perturbed by ε from the first

$$\chi(\varphi+\vartheta) = \begin{cases} x_c(\varphi)+\varepsilon & \text{for } \vartheta = 0 \\ x_c(\varphi+\vartheta) & \text{for } \vartheta \in [-\tau, 0) \end{cases}$$

Phase response curve:

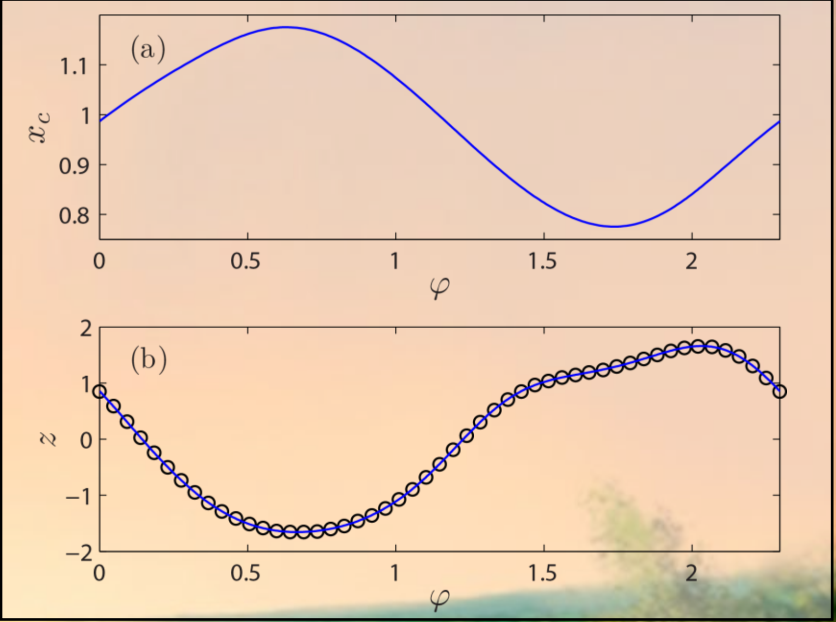
$$z(\varphi) = \Delta t / \varepsilon$$

$a = 2$

$b = 10$

$\tau = 0.7$

$\varepsilon = 10^{-5}$



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Example: Mackey-Glass equation

A perturbation with periodic external signal:

$$\frac{dx}{dt} = \frac{ax(t-\tau)}{1+x^b(t-\tau)} - x(t) + \varepsilon\psi(t)$$ here $$\psi(t) = \begin{cases} \sin(2\pi\upsilon t) \\ sign[\sin(2\pi\upsilon t)] \end{cases}$$

Frequency mismatch:

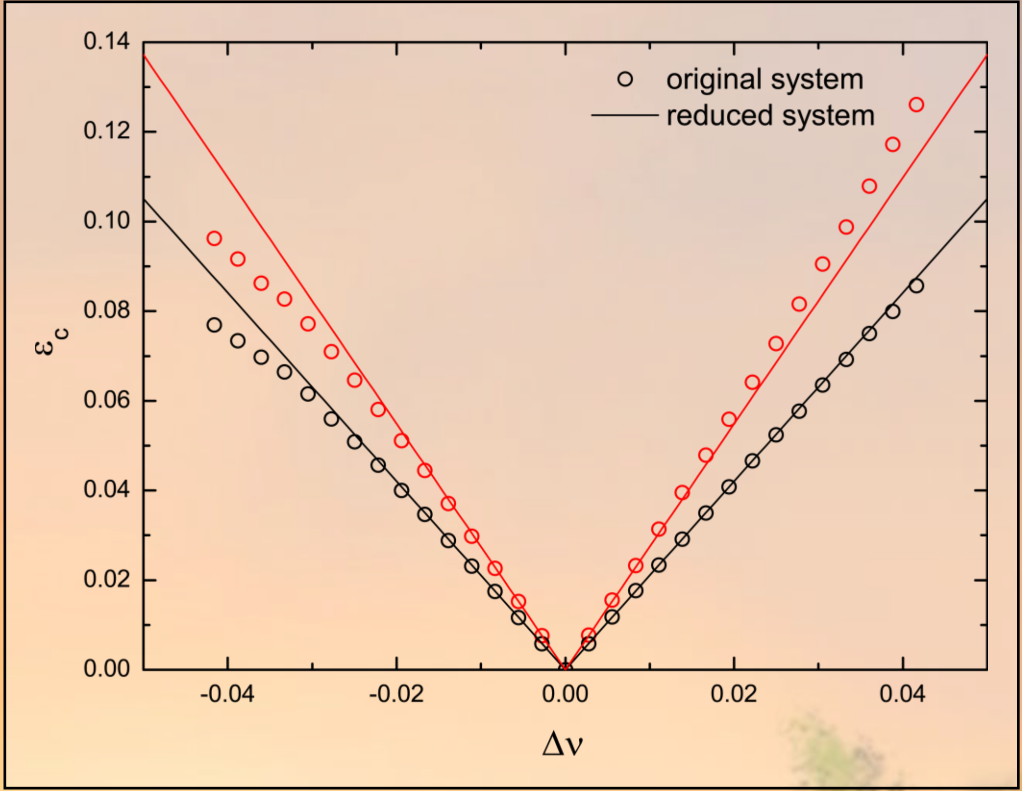
$$\Delta\upsilon = \upsilon - 1/T$$

Arnold's tongues:

$$a = 2$$

$$b = 10$$

$$\tau = 0.7$$



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Phase reduction of chaotic systems subject to a delayed feedback control

System with the stable limit cycle: $\boxed{\dot{x} = F\left(x(t)\right) + K\left[x(t-\tau) - x(t)\right]}$

The adjoint equation for PRC: $\boxed{\dot{z} = -A^T(t)z(t) - B^T(t+\tau)z(t+\tau)}$

$$\boxed{\begin{aligned} A(t) &= DF\left(x_c(t)\right) - K \\ B(t) &= K \end{aligned}}$$

The delay time τ is equal to PRC period, so the adjoint equation can be simplified to:

$\boxed{\dot{z} = -\left[DF\left(x_c\right)\right]^T z(t)}$ (Unstable in both directions)

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$\boxed{z^{(2)}(\varphi) = \alpha z^{(1)}(\varphi)}$

The coefficient of the proportionality α can be found from the initial condition:

$$\boxed{\alpha^{-1} = z^{(1)^T}(0)\dot{x}_c(0) + \int_{-\tau}^{0} z^{(1)^T}(\tau+\vartheta)K^{(2)}\dot{x}_c(\vartheta)d\vartheta}$$

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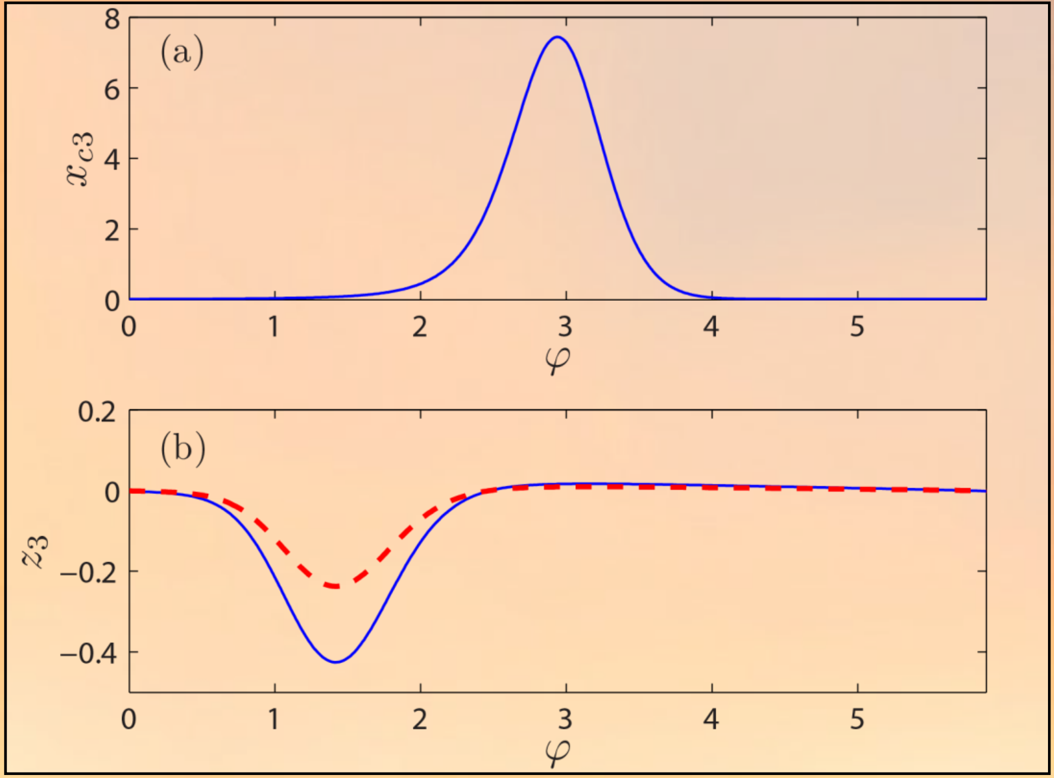
Example: Rossler system stabilized by DFC

$$\dot{x}_1 = -x_2 - x_3$$
$$\dot{x}_2 = x_1 + 0.2x_2 + K\left[x_2(t-\tau) - x_2(t)\right]$$
$$\dot{x}_3 = 0.2 + x_3(x_1 - 5.7)$$



K_1=0.15 and K_2=0.5

$\alpha \approx 0.558$

Conclusions

- A phase reduction method is applied to a general class of weakly perturbed time-delay systems exhibiting periodic oscillations
- An adjoint equation with an appropriate initial condition for the PRC of a time-delay system is derived by two methods
- The method is demonstrated numerically for the Mackey-Glass system as well as for a chaotic Rossler system subject a DFC
- The profile of the PRC of a periodic orbit stabilized by the DFC algorithm is independent of the control matrix

Acknowledgements

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