High-frequency analysis of periodically driven quantum system with slowly varying amplitude

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Main message

$$i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = H(\omega t, t)\left|\psi(t)\right\rangle$$

periodic dependence on the first argument: $H(\omega t + 2\pi, t) = H(\omega t, t)$

$\hbar\omega >>$ any other characteristic energies of the system

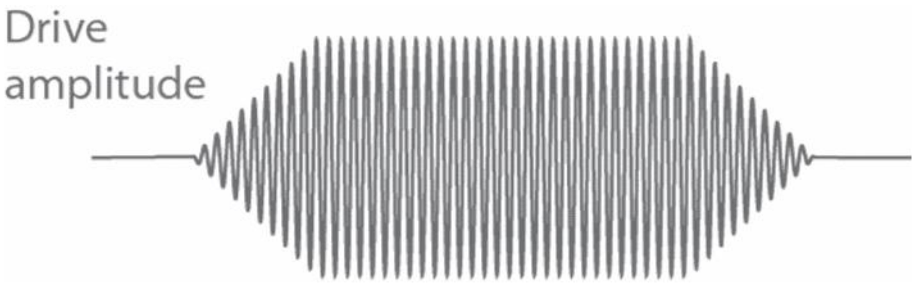
⇓

$$i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = H_{\text{eff}}(t)\left|\psi(t)\right\rangle$$

V. Novičenko, E. Anisimovas, G. Juzeliūnas, *Phys. Rev. A* **95**, 023615 (2017)

Motivation

shaken optical lattice



R. Desbuquois, M. Messer, F. Görg, K. Sandholzer, G. Jotzu, T. Esslinger, arXiv:1703.07767 (2017)

Extension of the space

Let us study whole family of the solutions:

$$i\hbar\frac{\partial}{\partial t}\left|\psi_\theta(t)\right\rangle = H(\omega t+\theta,t)\left|\psi_\theta(t)\right\rangle \qquad \theta\in[0,2\pi]$$

initial conditions:

$$\left|\psi_{\theta+2\pi}(t_{\text{init}})\right\rangle = \left|\psi_\theta(t_{\text{init}})\right\rangle$$

Hamiltonian $H(\omega t+\theta,t)$ acts on a Hilbert space $\mathcal{H}$

Introduce the space $\mathcal{T}$ of θ - periodic functions

Construct the space $\mathcal{L}=\mathcal{H}\otimes\mathcal{T}$

Apply unitary transformation $U=\exp\left[\omega t\frac{\partial}{\partial\theta}\right]$

⬇

$$K=U^\dagger HU-i\hbar U^\dagger\dot{U}=-i\hbar\omega\frac{\partial}{\partial\theta}+H(\theta,t)$$

⬇

$$K=\sum_{n=-\infty}^{\infty}\left|\overline{n}\right\rangle n\hbar\omega\mathbf{1}\left\langle\overline{n}\right|+\sum_{m,n=-\infty}^{\infty}\left|\overline{m}\right\rangle H^{(m-n)}(t)\left\langle\overline{n}\right|$$

Orthonormal basis of the space $\mathcal{T}$

$$e^{in\theta}\leftrightarrow\left|\overline{n}\right\rangle$$

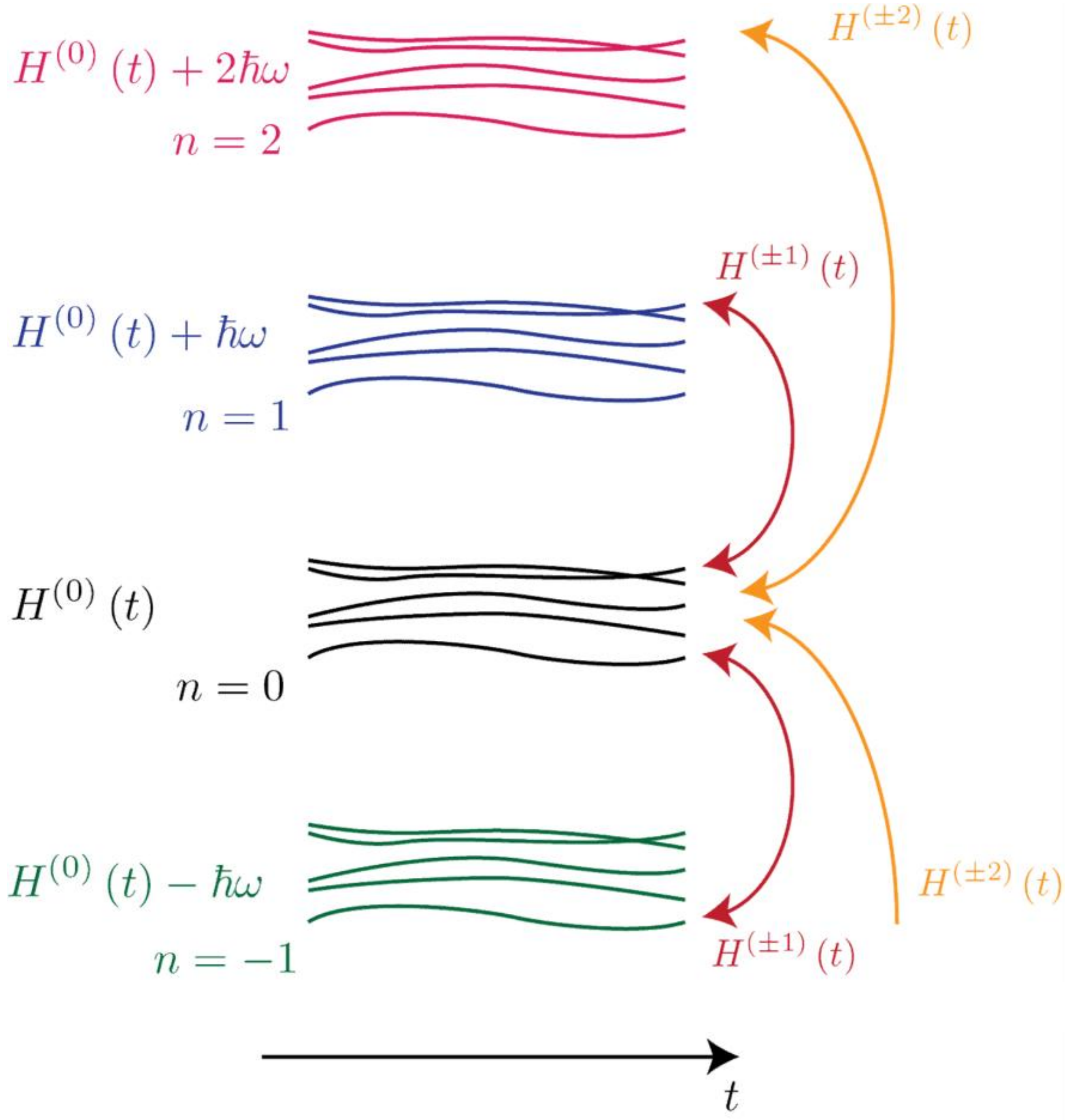
where the Fourier expansion of the Hamiltonian

$$H(\theta,t)=\sum_{l=-\infty}^{\infty}H^{(l)}(t)e^{il\theta}$$

“Kamiltonian” matrix

$$
K(t) = \begin{pmatrix}
 & \vdots & \vdots & \vdots & \vdots & \\
\cdots & H^{(0)}(t) - \hbar\mu\mathbf{1} & H^{(-1)}(t) & H^{(-2)}(t) & H^{(-3)}(t) & \cdots \\
\cdots & H^{(1)}(t) & H^{(0)}(t) & H^{(-1)}(t) & H^{(-2)}(t) & \cdots \\
\cdots & H^{(2)}(t) & H^{(1)}(t) & H^{(0)}(t) + \hbar\mu\mathbf{1} & H^{(-1)}(t) & \cdots \\
\cdots & H^{(3)}(t) & H^{(2)}(t) & H^{(1)}(t) & H^{(0)}(t) + 2\hbar\mu\mathbf{1} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{pmatrix}
$$

Floquet band structure of the "Kamiltonian" operator



Block diagonalization of the "Kamiltonian"

$$K_D(t) = D^\dagger(t)K(t)D(t) - i\hbar D^\dagger(t)\dot{D}(t) = \sum_{n=-\infty}^{\infty} |\bar{n}\rangle\left(H_{\text{eff}}(t) + n\hbar\mu\right)\langle\bar{n}|$$

$$K_D(t) = \begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \\ \cdots & H_{\text{eff}}(t) - \hbar\mu\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & H_{\text{eff}}(t) & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & H_{\text{eff}}(t) + \hbar\mu\mathbf{1} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & H_{\text{eff}}(t) + 2\hbar\mu\mathbf{1} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

High-frequency expansion

$$D(t) = \sum_{n=-\infty}^{\infty} |\bar{n}\rangle \mathbf{1} \langle \bar{n}| + D_{(1)}(t) + D_{(2)}(t) + O\left(\mu^{-3}\right) \qquad H_{\text{eff}}(t) = H_{\text{eff}(0)}(t) + H_{\text{eff}(1)}(t) + H_{\text{eff}(2)}(t) + O\left(\mu^{-3}\right)$$

$$D^{\dagger}(t)K(t)D(t) - i\hbar D^{\dagger}(t)\dot{D}(t) = \sum_{n=-\infty}^{\infty} |\bar{n}\rangle \left(H_{\text{eff}}(t) + n\hbar\mu\right) \langle \bar{n}|$$

$$H_{\text{eff}(0)} = H^{(0)}$$

$$H_{\text{eff}(1)} = \frac{1}{\hbar\mu} \sum_{m=1}^{\infty} \left[H^{(m)}, H^{(-m)}\right]$$

$$H_{\text{eff}(2)} = \frac{1}{(\hbar\mu)^2} \sum_{m\neq 0} \left\{ \frac{\left[H^{(-m)}, \left[H^{(0)}, H^{(m)}\right]\right] - i\hbar\left[H^{(-m)}, \dot{H}^{(m)}\right]}{2m^2} + \sum_{n\neq\{0,m\}} \frac{\left[H^{(-m)}, \left[H^{(m-n)}, H^{(n)}\right]\right]}{3mn} \right\}$$

Our original problem:

$$i\hbar \frac{\partial}{\partial t} |¤(t)\rangle = H(\mu t, t) |¤(t)\rangle$$

$$|¤(t_{\text{fin}})\rangle = U_{\text{Micro}}(\mu t_{\text{fin}}, t_{\text{fin}}) \underbrace{U_{\text{eff}}(t_{\text{fin}}, t_{\text{init}})}_{} U^{\dagger}_{\text{Micro}}(\mu t_{\text{init}}, t_{\text{init}}) |¤(t_{\text{init}})\rangle$$

$$i\hbar \frac{\partial}{\partial t} |¡(t)\rangle = H_{\text{eff}}(t) |¡(t)\rangle$$

High-frequency expansion

$$D(t)=\sum_{n=-\infty}^{\infty}|\overline{n}\rangle\mathbf{1}\langle\overline{n}|+D_{(1)}(t)+D_{(2)}(t)+O(\mu^{-3})\qquad H_{\text{eff}}(t)=H_{\text{eff}(0)}(t)+H_{\text{eff}(1)}(t)+H_{\text{eff}(2)}(t)+O(\mu^{-3})$$

$$D^{\dagger}(t)K(t)D(t)-i\hbar D^{\dagger}(t)\dot{D}(t)=\sum_{n=-\infty}^{\infty}|\overline{n}\rangle\left(H_{\text{eff}}(t)+n\hbar\mu\right)\langle\overline{n}|$$

$$H_{\text{eff}(0)}=\boxed{H^{(0)}}$$

$$H_{\text{eff}(1)}=\frac{1}{\hbar\mu}\sum_{m=1}^{\infty}\boxed{\left[H^{(m)},H^{(-m)}\right]}$$

$$H_{\text{eff}(2)}=\frac{1}{(\hbar\mu)^2}\sum_{m\neq 0}\left\{\frac{\boxed{\left[H^{(-m)},\left[H^{(0)},H^{(m)}\right]\right]}-\boxed{i\hbar\left[H^{(-m)},\dot{H}^{(m)}\right]}}{2m^2}+\sum_{n\neq\{0,m\}}\frac{\boxed{\left[H^{(-m)},\left[H^{(m-n)},H^{(n)}\right]\right]}}{3mn}\right\}$$

$= 0$

$\neq 0$

Our original problem:

$$i\hbar\frac{\partial}{\partial t}|¤(t)\rangle=H(\mu t,t)|¤(t)\rangle$$

$$|¤(t_{\text{fin}})\rangle=U_{\text{Micro}}(\mu t_{\text{fin}},t_{\text{fin}})\underbrace{U_{\text{eff}}(t_{\text{fin}},t_{\text{init}})}U^{\dagger}_{\text{Micro}}(\mu t_{\text{init}},t_{\text{init}})|¤(t_{\text{init}})\rangle$$

$$i\hbar\frac{\partial}{\partial t}|¡(t)\rangle=H_{\text{eff}}(t)|¡(t)\rangle$$

Spin in an oscillating magnetic field

The system Hamiltonian:

$$H(\mu t, t) = g_F \mathbf{F}\cdot\mathbf{B}(t)\cos(\mu t)$$

The non-zero Fourier components:

$$H^{(1)}(t) = H^{(-1)}(t) = \frac{g_F}{2}\mathbf{F}\cdot\mathbf{B}(t)$$

The effective Hamiltonian is non-zero only due to "slow" time derivative:

$$H_{\text{eff}}(t) = H_{\text{eff}(2)}(t) = \frac{-i\hbar}{(\hbar\mu)^2}\left[H^{(1)}, \dot{H}^{(1)}\right] = \mathbf{A}\cdot\dot{\mathbf{B}}$$

where we introduce the geometric matrix valued non-Abelian vector potential $\mathbf{A} = g_F^2(2\mu)^{-2}(\mathbf{F}\times\mathbf{B})$

The effective evolution:

$$U_{\text{eff}}(t_{\text{fin}}, t_{\text{init}}) = \mathcal{T}\exp\left[-\frac{i}{\hbar}\int_{t_{\text{init}}}^{t_{\text{fin}}}\mathbf{A}\cdot d\mathbf{B}(t)\right]$$

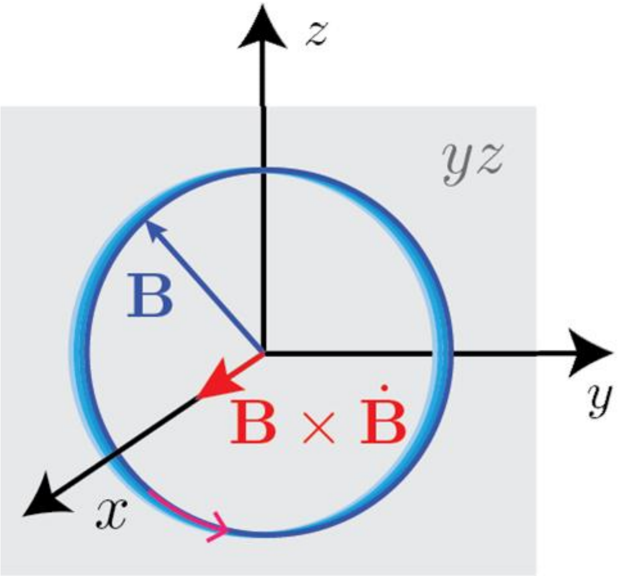
If $|\mathbf{B}(t)| = B = \text{const}$ and performs rotation in a plane by an angle ¨

$$U_{\text{eff}}(\mathbf{n}, ¨) = \exp\left[-\frac{i}{\hbar}¥_{¨}\,\mathbf{F}\cdot\mathbf{n}\right], \text{ where } ¥_{¨} = ¨\,\frac{g_F^2 B^2}{4\mu^2} \text{ and } \mathbf{n} = \frac{\mathbf{B}\times\dot{\mathbf{B}}}{|\mathbf{B}\times\dot{\mathbf{B}}|}$$

Numerical demonstration for a spin ½

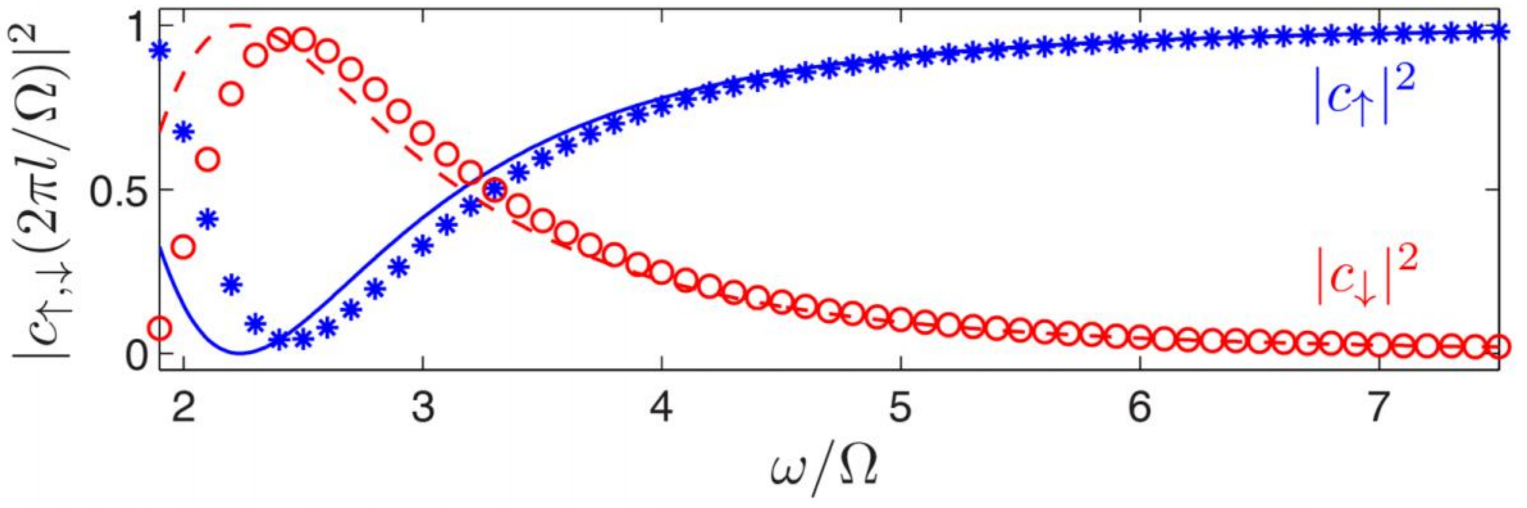
Magnetic field amplitude: $\mathbf{B}(t) = B\left[\mathbf{e}_z \cos(\Omega t) - \mathbf{e}_y \sin(\Omega t)\right]$

Wave function: $|¤(t)\rangle = c_\uparrow(t)|\uparrow\rangle + c_\downarrow(t)|\downarrow\rangle$



$c_\uparrow(t_{\text{init}}) = 1, \quad c_\downarrow(t_{\text{init}}) = 0$

$l = 10$ rotations



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